Integrable connections

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In this talk, I'll define the curvature of a connection, and define a connection to be *integrable* (or *flat* but this is potentially confusing) if its curvature vanishes. I'll then prove the main theorem, stating that on smooth manifolds, local systems are equivalent to vector bundles with integrable connections. This is a baby case of the much more general Riemann–Hilbert correspondence between D-modules and perverse sheaves. Finally I'll state the relative version of this theorem.

1 Review and an example

Let X be a complex manifold, V a holomorphic vector bundle on X (equivalently, a locally free sheaf of $\mathcal{O}_X$-modules). Recall the definition of a connection last time: it is a $\mathbb{C}$-linear map $\nabla : V \to \Omega^1_X \otimes V$, satisfying the Leibniz rule $\nabla(fs) = df \otimes s + f\nabla s$ for sections.

Intuition from differential geometry suggests that this should define a "parallel transport" identifying nearby fibers of the vector bundle. We saw last time that indeed it provides an identification of fibers over first order infinitesimal neighborhoods: $\Omega^1_X = \ker(\mathcal{O}_{X_1} \to \mathcal{O}_X)$, so ∇ gives a map $V \to \mathcal{O}_{X_1} \otimes V = (p_1)_*(p_2)^*V$, which by adjunction is a map $(p_1)^*V \to (p_2)^*V$ restricting to the identity on the diagonal $X \subset X_1$. Also recall from last time that we can inherit natural connections on direct sums, tensor products, duals, and sheaf homs of vector bundles with connections.

1.1 Example. Recall the example last time. Suppose we're working over a smooth curve. Locally, $X \subset \mathbb{C}$ and $V = \mathcal{O}^n$. Then a section of V is a bunch of functions $s_1, \ldots, s_n$. Let $M \in \Gamma(\underline{\mathrm{End}}V)$ be a matrix of holomorphic functions. If we want to solve $s' = Ms$, this is the same as $\nabla s = 0$ where $\nabla = d - M$. It is always possible to solve a linear system of ODEs locally.

On the other hand suppose we are on a higher dimensional manifold, say $X \subset \mathbb{C}^2$, and for simplicity just take $V = \mathcal{O}$. Any connection looks like $\nabla = d - \omega$, where $\omega = f(x,y)dx + g(x,y)dy$ is a 1-form. Then $\nabla s = 0$ is a system of PDEs

$$\frac{\partial s}{\partial x} = f(x,y),\ \frac{\partial s}{\partial y} = g(x,y)$$

which does not always have a solution (draw the square for example). The condition that guarantees a solution is $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, which is a nontrivial condition. So, for something like R–H correspondence to hold, we need to put some conditions on the connection. This is where the term "integrable" comes from. This should also be related to integrable systems.

2 The absolute case

Fix a connection ∇ on V. Recall that $\Omega^n_X = \wedge^n \Omega^1_X$ are sheaves of holomorphic differential forms. They carry a natural differential which forms the de Rham complex, which is a resolution of $\mathcal{O}_X$.

2.1 Definition. We can define $\nabla : \Omega_X^n \otimes V \to \Omega_X^{n+1} \otimes V$ by $\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^n \alpha \wedge \nabla s$.

This negative sign is from the Koszul sign convention. It is necessary for this to be well-defined:

2.2 Exercise. Check that $\nabla(f\alpha \otimes s) = \nabla(\alpha \otimes fs)$ for $f \in \Gamma(\mathcal{O}_X)$.

2.3 Exercise. Show that for $u_1 \in \Gamma(\Omega_X^{n_1} \otimes V_1)$, $u_2 \in \Gamma(\Omega_X^{n_2} \otimes V_2)$, $\nabla(u_1 \wedge u_2) = \nabla u_1 \wedge u_2 + (-1)^{n_1} u_1 \wedge \nabla u_2$. Here $u_1 \wedge u_2 \in \Gamma(\Omega_X^{n_1+n_2} \otimes V_1 \otimes V_2)$ naturally.

It is natural to ask whether this gives a differential on $\Omega_X^* \otimes V$.

2.4 Exercise. The map $\nabla\nabla : V \to \Omega_X^2 \otimes V$ is $\mathcal{O}_X$-linear.

Proof. $\nabla\nabla(fs) = \nabla(df \otimes s + f\nabla s) = -df \otimes \nabla s + df \otimes \nabla s + f\nabla\nabla s = f\nabla\nabla s$. □

2.5 Definition. The *curvature* R of ∇ is this map viewed as a section of $\Omega_X^2 \otimes \underline{\mathrm{End}} V$.

2.6 Exercise (Second Bianchi identity)**.** $\nabla R = 0$.

2.7 Exercise. $R(v,w)s = \nabla_v \nabla_w s - \nabla_w \nabla_v s - \nabla_{[v,w]} s$.

Proof. It's easy to verify $(\alpha \wedge \nabla s)(v,w) = \alpha(v)\nabla_w s - \alpha(w)\nabla_v s$. Now, let $\nabla s = \sum \alpha_i \otimes s_i$, then $\nabla\nabla s = \sum(d\alpha_i \otimes s_i + \alpha_i \wedge \nabla s_i)$, so applying this to (v,w) gives

$$\sum_i ((v(\alpha_i(w)) - w(\alpha_i(v)) - \alpha_i([v,w]))s_i + \alpha_i(v)\nabla_w s_i - \alpha_i(w)\nabla_v s_i) = \nabla_v \nabla_w s - \nabla_w \nabla_v s - \nabla_{[v,w]} s$$

as desired. □

2.8 Definition. A connection is *integrable* if $R \equiv 0$.

Here are three alternative ways of understanding integrability.

2.9 Remark. Local coordinates viewpoint. In local coordinates $z_1, \ldots, z_n$, and pick $e_1, \ldots, e_m$ basis of V, the map $\nabla : V \to \Omega_X^1 \otimes V$ is determined by $\nabla e_j = \Gamma_{ij}^k dz^i e_k$ (Christoffel symbols). The integrability condition is a collection of differential equations on Γ_{ij}^k. We can compute

$$\nabla\nabla e_k = \sum_{i<j} R_{ijk}^\ell (dz^i \wedge dz^j \otimes e_\ell)$$

where

$$R_{ijk}^\ell = \sum_s (\Gamma_{jk}^s \Gamma_{is}^\ell - \Gamma_{ik}^s \Gamma_{js}^\ell) + \frac{\partial \Gamma_{jk}^\ell}{\partial z_i} - \frac{\partial \Gamma_{ik}^\ell}{\partial z_j}.$$

This is the full Riemann curvature tensor as in differential geometry.

Another way to write it is Cartan's formalism: $\nabla = d + \omega$ where ω is a matrix of 1-forms, and $R = d\omega + \omega \wedge \omega$ is a matrix of 2-forms.

Flat spaces are not very interesting. But somehow in algebraic geometry we are more interested in flat connections.

2.10 Remark. Geometric viewpoint. Think about the vector bundle geometrically and fix local coordinates as above. For any section $s = (s_1, \ldots, s_m)$, We can compute $\nabla s = 0$ iff

$$\frac{\partial s_k}{\partial z_i} + \sum_j s_j \Gamma_{ij}^k = 0.$$

Thus, if we fix a point $x \in X$ and choose a point $x_0 \in V$ in its fiber (an initial condition), then the existence of this section is equivalent to the above system of PDEs having a solution. Suppose that there exists such a solution. Then the tangent bundle of this section (viewed as a submanifold of V) is given in coordinates by the image of $ds : TX \to TV$, which sends ∂_{z_i} to $\partial_{z_i} + \sum_k \frac{\partial s_k}{\partial z_i}\partial_{v_k}$. This motivates us to define $X_i = \partial_{z_i} - \sum_{j,k} v_j \Gamma^k_{ij}\partial_{v_k}$, for $1 \le i \le n$, which are vector fields on V which span a rank n subbundle W of TV. So $ds(TX)$ lands in W iff $\nabla s = 0$. One can compute $[X_i, X_j] = -\sum_{k,\ell} R^{\ell}_{ijk} v_k \partial_{v_\ell}$, so we conclude that $R \equiv 0$ iff $[X_i, X_j] = 0$ iff W is closed under brackets.

2.11 Remark. Crystalline viewpoint. Suppose X is smooth. Recall that ∇ can be interpreted as an isomorphism $p_1^* V \to p_2^* V$ which lifts the identity on $X \hookrightarrow X_1$, where $p_1, p_2 : X_1 \to X \times X \to X$. Now let X_1' be the first infinitesimal neighborhood of the diagonal in $X \times X \times X$. Then the isomorphism satisfies the cocycle condition on X_1' if and only if ∇ is integrable. This makes sense intuitively by the same picture (an infinitesimal triangle).

2.12 Theorem (Baby Riemann–Hilbert correspondence)**.** *There is an equivalence of categories*

$$\{\textit{finite dimensional local systems on } X\} \rightleftharpoons \{\textit{vector bundles on } X \textit{ with integrable connection}\}$$

given on the one hand by $L \mapsto V = L \otimes_{\mathbb{C}} \mathcal{O}_X$ together with the canonical connection *$\nabla(\ell \otimes f) = \ell \otimes df$, and on the other hand $V \mapsto L = \{s : \nabla s = 0\}$.*

We first prove a small lemma.

2.13 Proposition. *Let V be a vector bundle with any connection ∇. Then the sheaf L of horizontal sections of V forms a sheaf of $\mathbb{C}$-vector spaces which has finite dimensional stalks. Furthermore, the sets $X^{\le d} = \{x \in X : \dim L_x \le d\}$ are closed and $L|_{X^{\le d} \setminus X^{\le d-1}}$ are locally constant.*

Proof. For the first statement we can work locally and it suffices to show that given an initial value of a section at a point, there is at most one way of extending it to a germ. This is because the connection already tells you what the value has to be as an integral.

For the second statement, it is clear that $X \setminus X^{\le d}$ is open. Now let $x \in X^{\le d} \setminus X^{\le d-1}$. We can restrict to an open neighborhood of x which has exactly d independent horizontal sections. This gives a map of $\mathcal{O}$-modules $\mathcal{O}^d \to V$, which is an injection on the stalk at x, hence is a subbundle on some further open neighborhood of x, which then implies that the map $\underline{\mathbb{C}}^d \to L$ is an injection on stalks. Restricting to $X^{\le d} \setminus X^{\le d-1}$, it is an isomorphism on stalks, hence is an isomorphism. $\square$

Proof of theorem. 1) It is clear that the canonical connection is a connection. To see it is integrable, we can work locally and take coordinates $z_1, \ldots, z_n$ on X, and assume L is constant sheaf $\underline{\mathbb{C}}^m$. Then $\nabla(\ell \otimes df) = \sum \nabla_i(\ell \otimes \frac{\partial f}{\partial z_i} dz_i) = \sum_{i,j} \ell \otimes \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial z_i} dz_j \wedge dz_i = 0$.

2) Let V be a vector bundle with integrable connection. By the above lemma it suffices to show L has constant maximal dimension on stalks. In other words any initial condition determines some germ. Recall Frobenius's theorem which says that for any smooth real manifold M and pointwise independent vector fields $X_1, \ldots, X_n$, if their span is closed under brackets, then they arise from some foliation of M. This also works for complex manifolds, because a smooth submanifold of a complex manifold is a complex submanifold iff its tangent spaces are complex vector subspaces (by Newlander–Nirenberg, say). Applying this to $M = V$ and $X_1, \ldots, X_n$ as above, we see that there exists a section as desired. $\square$

2.14 Exercise. This correspondence preserves tensor product, duals and homs, and sends $\underline{\mathbb{C}}$ to $\mathcal{O}$.

2.15 Exercise. The chain complex $(\Omega_X^* \otimes V, \nabla)$, called the holomorphic de Rham complex with values (coefficients?) in V, forms a resolution of V.

2.16 Example (Gauss–Manin connection)**.** One way to get local systems is to take (co)homology. Suppose $f : E \to X$ is a proper map, where E, X are Hausdorff and X is locally compact. (Recall that this means $f^{-1}(\text{compact}) = \text{compact}$, or equivalently f is closed and fibers are compact.) Then for any sheaf F on E,

$$(R^i f_* F)|_x = H^i(f^{-1}(x), F|_{f^{-1}(x)}).$$

The LHS is the fiber, not the stalk. This is analogous to the cohomology and base change theorem. Now if f is smooth (a submersion) and E, X are complex manifolds, then Ehresmann's lemma tells us it is a fiber bundle, so $R^i f_* L$ are local systems for local systems L (suppose fibers have finite dimensional cohomology). It corresponds to the vector bundle $R^i f_* L \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X$, which is in fact isomorphic to $R^i f_*(\Omega^*_{X/S} \otimes_{\underline{\mathbb{C}}} L)$ (Deligne 2.28). The connection there is called the Gauss–Manin connection.

2.17 Example. For this example we want to take a family of elliptic curves. Let $X = \mathbb{P}^1 \backslash \{0, 1\}$ and E the family of smooth elliptic curves given by $y^2 z = x(x - z)(x - tz)$. (Another common choice is the Dwork family $x^3 + y^3 + z^3 - txyz$.) Then the $R^1 f_* \underline{\mathbb{C}}$ are rank 2 local systems on X. The 1-form $\omega = \frac{dx}{y}$ on E is a section of the local system, so it must satisfy some differential equation, simply because $\omega, \frac{d\omega}{dt}, \frac{d^2\omega}{dt^2}$ have to be linearly dependent fiberwise. You can just compute this directly and someone should tell me if there's a better way. This is called the Picard–Fuchs equation

$$t(1-t)\frac{d^2\omega}{dt^2} + (1-2t)\frac{d\omega}{dt} - \frac{1}{4},$$

which is a hypergeometric differential equation $a = b = \frac{1}{2}, c = 1$.

3 The relative case

All the above can be seen as the "absolute" case, over $\mathbb{C}$. All these make sense for schemes over an arbitrary base. Recall that a morphism of schemes is smooth (of relative dimension n) if the following equivalent conditions hold (Vakil 24.8.8):

- locally it looks like

  $$U \hookrightarrow \operatorname{Spec} A[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_m) \to \operatorname{Spec} A$$

  where the first map is an open immersion and $\det(\partial f_i/\partial x_j)_{1 \le i,j \le m}$ is nonzero on U.

- it is locally of finite presentation, flat of relative dimension n (all fibers have pure dimension n), and $\Omega_{X/Y}$ is locally free of rank n.

- it is locally of finite presentation, flat, and all fibers are smooth k-schemes of pure dimension n.

Now let $f : X \to S$ be a smooth morphism of schemes, and V a quasicoherent sheaf on X.

3.1 Definition. A relative connection on V is a $f^{-1}\mathcal{O}_S$-linear[1] map $\nabla : V \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} V$ satisfying Leibniz's rule. Similarly one can define the curvature $R \in \Omega^2_{X/S} \otimes \underline{\mathrm{End}} V$ and extend ∇ to a map $\Omega^n_{X/S} \otimes V \to \Omega^{n+1}_{X/S} \otimes V$, and define integrable connections.

[1]Deligne said $f^*\mathcal{O}_S$ here but I think it's a typo, and similar for a couple other places in the same section.

Same definition can be made for analytic spaces. Let $f : X \to S$ be a smooth morphism of analytic spaces. Recall this means that locally it is isomorphic to a projection $D \times S \to S$ where D is a polydisk.

3.2 Definition. A relative local system on V is a sheaf of $f^{-1}\mathcal{O}_S$-modules which is locally isomorphic to one of form $f^{-1}M$ for some coherent analytic sheaf M on S.

3.3 Theorem (Baby Riemann–Hilbert correspondence, relative case)**.** *There is an equivalence of categories*

$$\{\textit{relative local systems for } X\} \rightleftharpoons \{\textit{vector bundles on } X \textit{ with integrable relative connection}\}$$

given on the one hand by $L \mapsto V = L \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X$ *together with the* canonical connection $\nabla(\ell \otimes f) = \ell \otimes df$*, and on the other hand* $V \mapsto L = \{s : \nabla s = 0\}$*.*

I refer the reader to Deligne 2.23 for the proof.

3.4 Corollary. *For V integrable, the chain complex $(\Omega^*_{X/S} \otimes V, \nabla)$ forms a resolution of V.*