

# ALGEBRAIC DE RHAM COHOMOLOGY VIA STACKS

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ABSTRACT. We cover Chapter 2 of Bhatt's notes [Bha22] on Primstic  $F$ -gauges.

## 1. LINEAR ALGEBRA VIA STACKS

Let  $R$  be a commutative ring. We hope to express the category of graded  $R$ -modules and filtered  $R$ -modules using the language of stacks.

**1.1. Graded  $R$ -modules.** The derived category of graded  $R$ -modules is defined as  $\mathcal{D}_{gr}(R) := \text{Fun}(\mathbb{Z}, \mathcal{D}(R))$ , where  $\mathbb{Z}$  is considered a discrete category. Concretely, the objects of  $\mathcal{D}_{gr}(R)$  is just a collection of objects  $F(i) \in \mathcal{D}(R)$  indexed by integers  $i \in \mathbb{Z}$ . This is a symmetric monoidal category, with tensor product defined by:

$$(F \otimes G)(n) := \bigoplus_{i+j=n} F(i) \otimes G(j).$$

Now we can re-write  $\mathcal{D}_{gr}(R)$  using the language of stacks as follows. First, recall that  $B\mathbb{G}_m$  is the stack classifying line bundles on  $R$ -schemes, so it carries a tautological line bundle  $\mathcal{O}(1)$ .

**Proposition 1.2.** *There is an equivalence of monoidal categories*

$$\mathcal{D}_{gr}(R) \simeq \mathcal{D}_{qc}(B\mathbb{G}_m)$$

defined by

$$F \mapsto \bigoplus_{i \in \mathbb{Z}} F(i) \otimes_R \mathcal{O}(-i),$$

with inverse defined by

$$M \mapsto \left( i \mapsto R\Gamma(B\mathbb{G}_m, M(i)) \right).$$

Moreover, it fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{gr}(R) & \xrightarrow{\cong} & \mathcal{D}_{qc}(B\mathbb{G}_m) \\ & \searrow \text{Forg} & \swarrow \pi^* \\ & \mathcal{D}(R), & \end{array}$$

where the functor  $\text{Forg}$  forgets the grading (i.e., is  $M \mapsto \bigoplus_i M(i)$ ) and  $\pi$  is the map  $\text{Spec}(R) \rightarrow B\mathbb{G}_m$ .

**1.3. Filtered  $R$ -modules.** Next, we hope to provide a similar description for the category of filtered  $R$ -modules. In the non-derived setting, filtered  $R$ -modules were defined as follows:

**Definition 1.4.** A *filtered  $R$ -module* is a  $R$ -module  $F$  together with a sub-modules  $\text{Fil}^i F$  indexed by  $i \in \mathbb{Z}$  such that  $\text{Fil}^{i+1} F \subset \text{Fil}^i F$ . A filtered  $R$ -module is *exhaustive* if

$$F = \bigcup_{i \in \mathbb{Z}} \text{Fil}^i F.$$

Filtered  $R$ -modules can be visualized as a chain:

$$\dots \subset \mathrm{Fil}^{i+1} F \subset \mathrm{Fil}^i F \subset \mathrm{Fil}^{i-1} F \subset \dots .$$

We want to define the derived category of filtered  $R$ -modules  $\mathcal{D}_{fil}(R)$ . In the derived category, it *does not* make sense to talk about sub-modules, so we instead replace the inclusions  $\mathrm{Fil}^{i+1} M \hookrightarrow \mathrm{Fil}^i M$  by arbitrary maps. this gives the following:

**Definition 1.5.** The *derived category of filtered  $R$ -modules* is

$$\mathcal{D}_{fil}(R) := \mathrm{Fun}(\mathbb{Z}_{\geq}^{op}, \mathcal{D}(R)),$$

where  $\mathbb{Z}_{\geq}$  is the usual poset of integers considered as a category. We denote the value of a functor  $F$  at  $i \in \mathbb{Z}$  as  $\mathrm{Fil}^i F$ .

Now, a derived filtered  $R$ -module can be visualized as a chain:

$$\dots \rightarrow \mathrm{Fil}^{i+1} F \rightarrow \mathrm{Fil}^i F \rightarrow \mathrm{Fil}^{i-1} F \rightarrow \dots .$$

We give two sources of filtered  $R$ -modules:

**Example 1.6** (canonical filtration). There is a fully faithful embedding

$$\mathcal{D}(R) \hookrightarrow \mathcal{D}_{fil}(R),$$

which associates to  $K \in \mathcal{D}(R)$  the filtered  $R$ -module  $\tilde{K} \in \mathcal{D}_{fil}(R)$  given by

$$\mathrm{Fil}^i \tilde{K} := \tau^{\leq -i} K.$$

Here  $\mathrm{gr}^i \tilde{K} = (H^{-i} K)[i]$ . In fact, the essential image is exactly those  $F \in \mathcal{D}_{fil}(R)$  such that  $\mathrm{gr}^i F$  is concentrated in cohomological degree  $-i$  and is complete in the sense defined below.

**Example 1.7** (stupid filtration). There is a fully faithful functor

$$\mathrm{Ch}(R) \hookrightarrow \mathcal{D}_{fil}(R)$$

sending a chain complex  $K^\bullet$  of  $R$ -modules to

$$\mathrm{Fil}^i K^\bullet = K^{\geq i}.$$

Here  $\mathrm{gr}^i K^\bullet = K^i[-i]$ . In fact, the essential image is exactly those  $F \in \mathcal{D}_{fil}(R)$  such that  $\mathrm{gr}^i F$  is concentrated in cohomological degree  $i$ .

By analogy to non-derived filtered  $R$ -modules, we can define the following notions:

- The *underlying object* is  $\underline{F} := \mathrm{colim}_i \mathrm{Fil}^i F$ . For an non-derived exhaustive filtered  $R$ -module this is the usual notion of an underlying  $R$ -module.
- There is a symmetric monoidal structure on  $\mathcal{D}_{fil}(R)$  defined by

$$\mathrm{Fil}^n(F \otimes G) = \mathrm{colim}_{i+j \geq n} \mathrm{Fil}^i F \otimes \mathrm{Fil}^j G.$$

- We let  $\mathrm{gr}_{\mathrm{Fil}}^i F := \mathrm{Cone}(\mathrm{Fil}^{i+1} F \rightarrow \mathrm{Fil}^i F)$ . The construction

$$F \mapsto \mathrm{gr}_{\mathrm{Fil}}^* F = \bigoplus_i \mathrm{gr}_{\mathrm{Fil}}^i F$$

gives an exact colimit-preserving symmetric monoidal functor

$$\mathcal{D}_{fil}(R) \rightarrow \mathcal{D}_{gr}(R).$$

This is the derived analog of the classical construction

$$\mathrm{gr}_{\mathrm{Fil}}^* F = \bigoplus_i \mathrm{Fil}^i F / \mathrm{Fil}^{i+1} F.$$

- When  $F$  is a non-derived filtered  $R$ -module, the filtration is *complete* when

$$\bigcap_{i \in \mathbb{Z}} \text{Fil}^i F = 0.$$

Analogously, for  $F \in \mathcal{D}_{fil}(R)$ , we say  $F$  is *complete* when

$$\lim_i \text{Fil}^i F = 0.$$

Let  $\widehat{\mathcal{D}}_{fil}(R) \subset \mathcal{D}_{fil}(R)$  be the full subcategory of complete filtered  $R$ -modules. The inclusion has a left-adjoint given by

$$\widehat{F} := \text{Cone}(\text{Const}(\lim_i \text{Fil}^i F) \rightarrow F),$$

where for  $X \in \mathcal{D}(R)$ , we let  $\text{Const}(X)$  denote the constant functor  $\text{Fil}^i \text{Const}(X) = X$ .

- Given  $F \in \mathcal{D}_{fil}(R)$ , we can define  $F\{n\}$  to be shift by  $n$ , i.e.,

$$\text{Fil}^i(F\{n\}) := \text{Fil}^{i+n} F.$$

There is a natural map  $F\{1\} \rightarrow F$ , given by the map

$$\text{Fil}^i(F\{1\}) = \text{Fil}^{i+1} F \rightarrow \text{Fil}^i F,$$

such that

$$\text{Fil}^i(F/F\{1\}) = \text{gr}^i F.$$

- There are two natural  $t$ -structures on  $\mathcal{D}_{fil}(R)$ :
  - The standard  $t$ -structure:  $F \in \mathcal{D}_{fil}(R)$  is connective (resp., co-connective) when  $\text{Fil}^i F$  is connective (resp., co-connective) for each  $i$
  - The Beilinson  $t$ -structure:  $F \in \mathcal{D}_{fil}(R)$  is connective (resp., co-connective) when  $\text{gr}^i F \in \mathcal{D}^{\leq i}$  (resp.,  $\text{gr}^i F \in \mathcal{D}^{\geq i}$ ) for each  $i \in \mathbb{Z}$ . By Example 1.7, the stupid filtration gives an equivalence between  $\text{Ch}(R)$  and the heart of the Beilinson  $t$ -structure.

We ultimately hope to prove an equivalence between  $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$  and  $\mathcal{D}_{fil}(R)$ . Here, we let  $\mathbb{A}^1 = \text{Spec } R[t]$  and let the  $\mathbb{G}_m$ -action give  $t$  degree 1. Thus, there is an equivalence  $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathcal{D}_{gr}(R[t])$ , where  $\mathcal{D}_{gr}(R[t])$  is the derived category of graded  $R[t]$ -modules. First, let us recall what the stack  $\mathbb{A}^1/\mathbb{G}_m$  classifies:

**Remark 1.8.** Given a scheme  $T$ , the groupoid  $\mathbb{A}^1/\mathbb{G}_m(T)$  classifies  $\mathbb{G}_m$ -torsors  $T' \rightarrow T$  together with a  $\mathbb{G}_m$ -equivariant map  $T' \rightarrow \mathbb{A}^1$ . A  $\mathbb{G}_m$ -torsor must be of the form

$$T' = \text{Spec}(\bigoplus_{i \in \mathbb{Z}} \mathcal{L}^{-i})$$

for a line bundle  $\mathcal{L}$  on  $T$ , and a map  $T' \rightarrow \mathbb{A}^1$  is equivalent to a  $\mathcal{O}_T$ -linear map  $\mathcal{O}_T \rightarrow \mathcal{L}^{-1}$ . Thus

$$\mathbb{A}^1/\mathbb{G}_m(T) \simeq \{\text{a line bundle } \mathcal{L} \text{ on } T, \text{ with a } \mathcal{O}_T\text{-linear map } \mathcal{L} \rightarrow \mathcal{O}_T\}.$$

In particular, there is a universal pair  $t: \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}(-1) \rightarrow \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}$  over  $\mathbb{A}^1/\mathbb{G}_m$ . As graded  $R[t]$ -modules, this is the inclusion

$$tR[t] \subset R[t]$$

of graded  $R[t]$ -modules. The vanishing locus of  $t$  is the Cartier divisor  $B\mathbb{G}_m \subset \mathbb{A}^1/\mathbb{G}_m$ . Moreover,  $\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}(t)|_{B\mathbb{G}_m} \simeq \mathcal{O}_{B\mathbb{G}_m}(-1)$ , since the graded  $R[t]$ -module  $tR[t]/t^2R[t]$  is a copy of  $R$  in degree 1.

Now, the main theorem is:

**Theorem 1.9.** *There is an equivalence of symmetric monoidal categories*

$$\text{Rees}: \mathcal{D}_{\text{fil}}(R) \simeq \mathcal{D}_{\text{qc}}(\mathbb{A}^1/\mathbb{G}_m)$$

defined by sending  $F \in \mathcal{D}_{\text{fil}}(R)$  to the graded  $R[t]$ -module

$$\text{Rees}(F) := \bigoplus_{i \in \mathbb{Z}} \text{Fil}^i F \cdot t^{-i}.$$

It has the following properties:

- (1) Rees is  $t$ -exact with the standard  $t$ -structures.
- (2) There is a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{\text{fil}}(R) & \xrightarrow[\simeq]{\text{Rees}} & \mathcal{D}_{\text{qc}}(\mathbb{A}^1/\mathbb{G}_m) \\ & \searrow \text{Forg} & \swarrow j^* \\ & \mathcal{D}(R), & \end{array}$$

where  $j: \text{Spec}(R) = \mathbb{G}_m/\mathbb{G}_m \rightarrow \mathbb{A}^1/\mathbb{G}_m$  is the standard open immersion and *Forg* forgets the filtration (i.e., takes the underlying module.)

- (3) Restriction to the Cartier divisor  $i: B\mathbb{G}_m \rightarrow \mathbb{A}^1/\mathbb{G}_m$  corresponds to passing to the associated graded, up to a change of sign. More precisely, for  $F \in \mathcal{D}_{\text{fil}}(R)$  and  $i \in \mathbb{Z}$ ,

$$i^* \text{Rees}(F) = \bigoplus_{i \in \mathbb{Z}} \text{gr}_{\text{Fil}}^i F \otimes \mathcal{O}(i),$$

or, equivalently,

$$\text{gr}_{\text{Fil}}^i F \simeq R\Gamma(B\mathbb{G}_m, i^* \text{Rees}(F)(-i)).$$

- (4)  $F \in \mathcal{D}_{\text{fil}}(R)$  is complete as a filtered  $R$ -module if and only if  $\text{Rees}(F) \in \mathcal{D}_{\text{qc}}(\mathbb{A}^1/\mathbb{G}_m)$  is derived  $t$ -complete. Here,  $M \in \mathcal{D}_{\text{qc}}(\mathbb{A}^1/\mathbb{G}_m)$  is derived  $t$ -complete when the derived limit of the diagram

$$\dots \xrightarrow{t} M \xrightarrow{t} M \xrightarrow{t} M$$

is zero.

- (5) For any  $F \in \mathcal{D}_{\text{fil}}(R)$  there is an isomorphism  $\text{Rees}(F\{n\}) \simeq \text{Rees}(F) \otimes \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}(-n)$ .

*Proof Sketch.* Given an object  $M \in \mathcal{D}_{\text{qc}}(\mathbb{A}^1/\mathbb{G}_m)$ , we can consider the filtered  $R$ -module given by taking  $R\Gamma(\mathbb{A}^1/\mathbb{G}_m, -)$  of the diagram

$$\dots \xrightarrow{t} M \otimes_{\mathcal{O}} \mathcal{O}(i-1) \xrightarrow{t} M \otimes_{\mathcal{O}} \mathcal{O}(i) \xrightarrow{t} M \otimes_{\mathcal{O}} \mathcal{O}(i+1) \xrightarrow{t} \dots$$

In the language of graded  $R[t]$ -modules, given  $M = \bigoplus_i M(i) \in \mathcal{D}_{\text{gr}}(R[t])$ , we can take

$$\text{Fil}^i(\text{Rees}^{-1} M) = M(-i)$$

with transition maps  $t: M(-i) \rightarrow M(-i+1)$ . □

**Remark 1.10.** Any perfect complex  $M \in \text{Perf}(\mathbb{A}^1/\mathbb{G}_m)$  is derived  $t$ -complete. Indeed, it suffices to check this when  $M = \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}$ , which is essentially the fact that  $k[t] \simeq \lim k[t]/t^n$  as graded vector spaces.

**Remark 1.11** (Vector bundles on  $\mathbb{A}^1/\mathbb{G}_m$ ). Under the Rees equivalence, the category  $\text{Vect}(\mathbb{A}^1/\mathbb{G}_m)$  of vector bundles on  $\mathbb{A}^1/\mathbb{G}_m$  is identified with the category of pairs  $(M, F^*)$  where  $M$  is a finite projective  $R$ -module and  $F^*$  is a finite exhaustive filtration on  $M$  (in the non-derived sense) such that  $\text{gr}_F^i M$  is finite projective for all  $i$ .

**Remark 1.12** (Canonical filtrations). The composition of functors from Example 1.6 and the theorem gives

$$\mathcal{D}(R) \hookrightarrow \mathcal{D}_{fil}(R) \simeq \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m).$$

The essential image consists of objects  $M \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$  which are complete and such that  $\mathcal{H}^i(M)(i)$  is constant, i.e., pulled back from  $\text{Spec}(R)$ , for all  $i$ .

**Definition 1.13.** A *filtered stack* is a stack  $\mathfrak{X}$  together with a morphism  $f: \mathfrak{X} \rightarrow \mathbb{A}^1/\mathbb{G}_m$ .

**Remark 1.14.** A filtered stack can be viewed as a filtration on the stack

$$\underline{\mathfrak{X}} := f^{-1}(\mathbb{G}_m/\mathbb{G}_m)$$

with associated graded

$$\text{Gr}(\underline{\mathfrak{X}}) := f^{-1}(B\mathbb{G}_m).$$

Assuming  $f_*$  preserves quasi-coherence (e.g., when  $f$  is representable qcqs), for any  $M \in \mathcal{D}_{qc}(\underline{\mathfrak{X}})$ , the pushforward

$$f_*M \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathcal{D}_{fil}(R)$$

is a filtration on  $R\Gamma(\underline{\mathfrak{X}}, M)$  with associated graded  $R\Gamma(\text{Gr}(\underline{\mathfrak{X}}), M)$ .

### 1.15. Endomorphisms and $B\widehat{\mathbb{G}}_a$ .

**Definition 1.16.** Let  $\widehat{\mathbb{G}}_a \subset \mathbb{G}_a$  be the formal completion at 0; the functor of points is  $\widehat{\mathbb{G}}_a(S) = \text{Nil}(S)$  for any  $R$ -algebra  $S$ .

Then we have the proposition:

**Proposition 1.17.** *Let  $R$  be a commutative  $\mathbb{Q}$ -algebra. There is an equivalence of symmetric monoidal categories*

$$\Phi: \mathcal{D}_{qc}(B\widehat{\mathbb{G}}_a) \simeq \mathcal{D}(R[t]),$$

where  $\mathcal{D}(R[t])$  is a symmetric monoidal category under convolution, i.e., for  $M, N \in \mathcal{D}(R[t])$  the convolution  $M \star N$  has underlying  $R$ -module  $M \otimes_R N$  and  $t$  acts via  $t_M \otimes 1_N + 1_M \otimes t_N$ . The functor  $\Phi$  has properties:

(1) *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{D}_{qc}(B\widehat{\mathbb{G}}_a) & \xrightarrow[\simeq]{\Phi} & \mathcal{D}(R[t]) \\ & \searrow \pi^* & \swarrow \text{Forg} \\ & \mathcal{D}(R) & \end{array}$$

where  $\pi: \text{Spec}(R) \rightarrow B\widehat{\mathbb{G}}_a$  is the standard map and  $\text{Forg}$  forgets the action of  $t$ .

(2)  $\Phi$  sends  $\mathcal{O}_{B\widehat{\mathbb{G}}_a}$  to  $R[t]/(t) \simeq R$ . Thus, for  $M \in \mathcal{D}_{qc}(B\widehat{\mathbb{G}}_a)$ , there is a natural isomorphism

$$R\Gamma(B\widehat{\mathbb{G}}_a, M) \simeq \text{RHom}_{k[t]}(k, \Phi(M)) \simeq \text{Fib}(\Phi(M) \xrightarrow{t} \Phi(M)).$$

In particular,  $R\Gamma(B\widehat{\mathbb{G}}_a, -)$  has cohomological dimension 1.

*Proof.* Write  $X$  for the coordinate on  $\widehat{\mathbb{G}}_a$  that is  $\mathbb{G}_m$ -equivariantly dual to  $t$ . Then for  $M \in \mathcal{D}(R[t])$ , the corresponding quasi-coherent sheaf on  $B\widehat{\mathbb{G}}_a$  is given by the co-action map

$$c: M \rightarrow M[[X]]$$

$$m \mapsto \exp(tX)m := \sum_{i \geq 0} t^i(m) \frac{X^i}{i!}.$$

Moreover, the  $R[t]$ -module structure can be recovered as the coefficient of  $X$ . □

**Example 1.18.** The inclusion  $\widehat{\mathbb{G}}_a \subset \mathbb{G}_a$  gives a  $k[t]$ -module structure on  $\mathcal{O}(\mathbb{G}_a) = k[X]$ , which is simply  $t = \frac{d}{dX}$ .

In other words,  $\widehat{\mathbb{G}}_a$ -representations are equivalent to modules with an endomorphism.

In fact, Proposition 1.17 can be upgraded to work in families. Given a finite projective  $R$ -module  $E$ , consider the associated vector bundle  $\mathbf{V}(E)$ . We can analogously define  $\widehat{\mathbf{V}}(E)$ . Then we have:

**Proposition 1.19.** *Let  $R$  be a commutative  $\mathbb{Q}$ -algebra and let  $E$  be a finite projective  $R$ -module. Then there is a natural equivalence of symmetric monoidal categories*

$$\mathcal{D}_{qc}(\widehat{B\mathbf{V}}(E)) \simeq \mathcal{D}_{qc}(\mathbf{V}(E^\vee))$$

where  $\mathcal{D}_{qc}(\mathbf{V}(E^\vee))$  is given the convolution product. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{qc}(\widehat{B\mathbf{V}}(E)) & \xrightarrow[\simeq]{\Phi} & \mathcal{D}_{qc}(\mathbf{V}(E^\vee)) \\ & \searrow \pi^* & \swarrow s_* \\ & \mathcal{D}(R) & \end{array}$$

where as usual  $\pi: \text{Spec}(R) \rightarrow \widehat{B\mathbf{V}}(E)$  is the tautological map, and  $s$  is the structure map  $\mathbf{V}(E^\vee) \rightarrow \text{Spec}(R)$ .

**Remark 1.20.** We can use the Proposition to compute the cohomology of  $\widehat{\mathbf{V}}(E)$ -representations. Recall that an object  $M \in \mathcal{D}_{qc}(\widehat{B\mathbf{V}}(E))$  can be regarded as a  $\widehat{\mathbf{V}}(E)$ -representation on  $\pi^*M \in \mathcal{D}(R)$ . By the proposition,  $\pi^*M$  carries a natural action of  $S = \text{Sym}_R^*(E)$ , and

$$R\Gamma(\widehat{B\mathbf{V}}(E), M) := \text{RHom}_{\widehat{B\mathbf{V}}(E)}(\mathcal{O}, M) \simeq \text{RHom}_S(R, \pi^*M).$$

The derived Hom can be computed using the Koszul resolution of  $R$ .

We need a global version of the Remark 1.20:

**Remark 1.21.** Suppose we have a qcqs morphism  $f: Y \rightarrow Z$  of characteristic 0 schemes, a line bundle  $\mathcal{L}$  on  $Z$ , and a  $Z$ -linear action of  $G = \widehat{\mathbf{V}}(\mathcal{L})$  on  $Y$ . Then we have a recipe to compute pushforwards along  $f_G: Y/G \rightarrow Z$ . Consider the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \pi_Y \downarrow & & \downarrow \pi_Z \\ Y/G & \xrightarrow{\tilde{f}} & BG. \end{array}$$

The horizontal map is qcqs, so the pushforward along the map preserves quasi-coherence. Moreover, given  $M \in \mathcal{D}_{qc}(Y/G)$ , flat base change shows

$$\pi_Z^* R\tilde{f}_* M \simeq Rf_* \pi_Y^* M.$$

Pushing forward  $R\tilde{f}_* M$  along the structure map  $g: BG \rightarrow Z$  and using Remark 1.20, we learn that  $Rf_{G,*} M \simeq Rg_* R\tilde{f}_* M$  sits in a fiber sequence

$$Rf_{G,*} M \rightarrow Rf_* \pi_Y^* M \rightarrow Rf_* \pi_Y^* M \otimes \mathcal{L}^{-1}.$$

Thus,  $Rf_{G,*} M$  is quasi-coherent.

Analogously, there are equivalences:

$$\begin{aligned} \mathcal{D}_{qc}(B\mathbb{G}_a) &\simeq \mathcal{D}_{qc}(\widehat{\mathbb{G}}_a) \\ \mathcal{D}_{qc}(\widehat{B\mathbf{V}}(E)) &\simeq \mathcal{D}_{qc}(\widehat{\mathbf{V}}(E^\vee)), \end{aligned}$$

swapping the role of  $\mathbb{G}_a$  and  $\widehat{\mathbb{G}}_a$ .

We will use the following variant of Serre vanishing:

**Lemma 1.22.** *Let  $R$  be any commutative ring. Then*

$$R\Gamma_{\text{ét}}(\text{Spec}(R), \widehat{\mathbb{G}}_a) \simeq \text{Nil}(R)[0].$$

*Proof.* It suffices to check that for any étale cover  $R \rightarrow S$  with Čech nerve  $R \rightarrow S^\bullet$  we have

$$\text{Nil}(R) \simeq \lim \text{Nil}(S^\bullet).$$

Since  $R \rightarrow S$  is étale,  $\text{Nil}(R) \otimes_R S^\bullet \simeq \text{Nil}(S^\bullet)$  so

$$\lim \text{Nil}(S^\bullet) \simeq \lim \text{Nil}(R) \otimes_R S^\bullet \simeq \text{Nil}(R)$$

by fpqc descent for quasi-coherent sheaves.  $\square$

## 2. DE RHAM COHOMOLOGY IN CHARACTERISTIC 0 VIA STACKS

In this section, we work over a ground field  $k$  of characteristic 0.

**Definition 2.1.** The scheme  $\mathbb{G}_a$  is naturally a ring scheme, and the subfunctor  $\widehat{\mathbb{G}}_a \subset \mathbb{G}_a$  is an ideal group scheme. Thus, the quotient sheaf  $\mathbb{G}_a^{dR} := \mathbb{G}_a / \widehat{\mathbb{G}}_a$  is a sheaf of rings, and for any ring  $R$ ,

$$\mathbb{G}_a^{dR}(R) = R_{\text{red}}.$$

**Remark 2.2.** Of course,  $\mathbb{G}_a(R) / \widehat{\mathbb{G}}_a(R) = R / \text{Nil}(R) = R_{\text{red}}$ . The fact that even upon sheafification we have  $\mathbb{G}_a^{dR}(R) = R_{\text{red}}$  is due to Lemma 1.22.

Now, using the ring stack  $\mathbb{G}_a^{dR}$ , we can define the de Rham space:

**Definition 2.3.** For any  $k$ -scheme  $X$ , let the *de Rham space*  $X^{dR}$  be the functor on finite-type  $k$ -algebras given by

$$X^{dR}(R) := X(\mathbb{G}_a^{dR}(R)) = X(R_{\text{red}}).$$

**Remark 2.4.** In general, there is a natural map  $X \rightarrow X^{dR}$  induced by the quotient map  $\mathbb{G}_a \rightarrow \mathbb{G}_a^{dR}$ . When  $X$  is smooth, this map  $X \rightarrow X^{dR}$  is a surjection of étale sheaves, by the infinitesimal lifting property of smoothness. For any  $k$ -algebra  $T$ , we claim  $X(T) \rightarrow X(T^{red})$  is surjective. But the infinitesimal lifting property states that in the diagram

$$\begin{array}{ccc} \text{Spec}(T^{red}) & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spec}(T) & \longrightarrow & \text{Spec}(k), \end{array}$$

there exists a lifting  $\text{Spec}(T) \rightarrow X$ , which is exactly what we want.

Now our goal is to show:

**Theorem 2.5** (de Rham cohomology via  $X^{dR}$ ). *For a smooth  $k$ -scheme  $X$ , there is a natural identification*

$$R\Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) \simeq R\Gamma(X, \Omega_{X/k}^\bullet).$$

*Under this isomorphism, pulling back along  $X \rightarrow X^{dR}$  corresponds to the projection*

$$\text{gr}_H^0 R\Gamma(X, \Omega_{X/k}^\bullet) \simeq R\Gamma(X, \mathcal{O}_X)$$

*given by the Hodge filtration.*

To prove the theorem, we construct a filtration on  $R\Gamma(X^{dR}, \mathcal{O}_{X^{dR}})$ , and, in fact, on  $X^{dR}$ . Recall that filtering  $X^{dR}$  means finding a stack  $\mathfrak{X} \rightarrow \mathbb{A}^1 / \mathbb{G}_m$  such that  $\mathfrak{X}|_{\mathbb{G}_m / \mathbb{G}_m} \simeq X^{dR}$ .

**Definition 2.6.** Consider the universal effective Cartier divisor  $t: \mathcal{O}(-1) \rightarrow \mathcal{O}$  on the stack  $\mathbb{A}^1/\mathbb{G}_m$ . Passing to the associated vector bundle schemes, we have a morphism

$$d: \mathbf{V}(\widehat{\mathcal{O}(-1)}) \xrightarrow{t} \mathbf{V}(\mathcal{O}) = \mathbb{G}_a.$$

over  $\mathbb{A}^1/\mathbb{G}_m$ . Now, the stack quotient

$$\mathbb{G}_a^{dR,+} = \text{Cone}(\mathbf{V}(\widehat{\mathcal{O}(-1)}) \rightarrow \mathbb{G}_a)$$

becomes a 1-truncated animated  $\mathbb{G}_a$ -algebra over  $\mathbb{A}^1/\mathbb{G}_m$ . In other words, if a map  $\text{Spec}(R) \rightarrow \mathbb{A}^1/\mathbb{G}_m$  is given by  $(L \in \text{Pic}(R), L \rightarrow R)$ , then

$$\mathbb{G}_a^{dR,+}(\text{Spec}(R) \rightarrow \mathbb{A}^1/\mathbb{G}_m) = \text{Cone}(\text{Nil}(R) \otimes_R L \rightarrow R).$$

**Remark 2.7.** There are isomorphisms

$$\begin{aligned} \mathbb{G}_a^{dR,+}|_{\mathbb{G}_m/\mathbb{G}_m} &\simeq \mathbb{G}_a^{dR} \\ \mathbb{G}_a^{dR,+}|_{B\mathbb{G}_m} &\simeq \mathbb{G}_a^{\text{Hodge}}, \end{aligned}$$

where

$$\mathbb{G}_a^{\text{Hodge}} := \mathbb{G}_a \oplus \mathbf{V}(\widehat{\mathcal{O}(-1)})[1],$$

i.e.,

$$\mathbb{G}_a^{\text{Hodge}}(\text{Spec}(R) \rightarrow B\mathbb{G}_m) = R \oplus \text{Nil}(R) \otimes_R L[1].$$

Indeed, over  $\mathbb{G}_m/\mathbb{G}_m$  the map  $L \rightarrow R$  is an isomorphism, so

$$\text{Cone}(\text{Nil}(R) \otimes_R L \rightarrow R) \simeq \text{Cone}(\text{Nil}(R) \rightarrow R) = R_{\text{red}}$$

and over  $\mathbb{A}^1/\mathbb{G}_m$ , the map  $L \rightarrow R$  is zero so

$$\text{Cone}(\text{Nil}(R) \otimes_R L \rightarrow R) \simeq R \oplus \text{Nil}(R) \otimes_R L[1]$$

Now, we can define the filtered de Rham stack:

**Definition 2.8.** For a smooth  $k$ -scheme  $X$ , the *filtered de Rham space* is the map  $X^{dR,+} \rightarrow \mathbb{A}^1/\mathbb{G}_m$  whose functor of points is

$$X^{dR,+}(\text{Spec}(R) \rightarrow \mathbb{A}^1/\mathbb{G}_m) = X(\mathbb{G}_a^{dR,+}(R)),$$

where the right-hand side is the groupoid of maps  $\text{Spec}(\mathbb{G}_a^{dR,+}(R)) \rightarrow X$  in derived algebraic geometry. The fiber

$$X^{\text{Hodge}} := X^{dR,+} \times_{\mathbb{A}^1/\mathbb{G}_m} B\mathbb{G}_m$$

is called the *Hodge stack* of  $X$ , so the functor  $X^{\text{Hodge}}$  on  $B\mathbb{G}_m$ -schemes is given by

$$X^{\text{Hodge}}(\text{Spec}(R) \rightarrow B\mathbb{G}_m) = X(\mathbb{G}_a^{\text{Hodge}}(R)).$$

The filtered de Rham stack recovers  $X^{dR}$  over  $\mathbb{G}_m/\mathbb{G}_m$ :

$$X^{dR} \simeq X^{dR,+} \times_{\mathbb{A}^1/\mathbb{G}_m} \mathbb{G}_m/\mathbb{G}_m.$$

**Remark 2.9.** Generalizing Remark 2.4, the quotient maps  $\mathbb{G}_a \rightarrow \mathbb{G}_a^{dR}$ ,  $\mathbb{G}_a \rightarrow \mathbb{G}_a^{\text{Hodge}}$ , and  $\mathbb{G}_a \rightarrow \mathbb{G}_a^{dR,+}$  induce maps

$$\begin{aligned} X &\rightarrow X^{dR} \\ X \times B\mathbb{G}_m &\rightarrow X^{\text{Hodge}} \\ X \times \mathbb{A}^1/\mathbb{G}_m &\rightarrow X^{dR,+}. \end{aligned}$$

When  $X$  is smooth, all of these are surjections of étale sheaves.

Now, Theorem 2.5 follows from the stronger theorem:



**Theorem 2.10** (Hodge-filtered de Rham cohomology via  $X^{dR,+}$ ). *For  $X/k$  a smooth variety, let  $\pi_X : X^{dR,+} \rightarrow \mathbb{A}^1/\mathbb{G}_m$  be the structure map. Then*

$$\mathcal{H}_{dR,+}(X) := R\pi_* \mathcal{O}_{X^{dR,+}}$$

*is quasi-coherent and complete, and the corresponding filtered object of  $\widehat{\mathcal{D}}_{fil}(k)$  identifies with the Hodge-filtered de Rham cohomology  $\mathrm{Fil}_H^* R\Gamma(X, \Omega_{X/k}^\bullet)$ .*

In fact, what we prove is that

$$U \mapsto \mathcal{H}_{dR,+}(U) \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$$

can be regarded as a Zariski sheaf  $\mathcal{F}$  on  $X$  valued in  $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$ . The category

$$\mathrm{Shv}(X, \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)) \simeq \mathrm{Shv}(X, \mathcal{D}_{fil}(k))$$

carries a Beilinson  $t$ -structure, whose heart is the abelian category of chain complexes of sheaves of  $k$ -modules on  $X$ . We show that  $U \mapsto \mathcal{H}_{dR,+}(U)$  lies in the heart of the  $t$ -structure, and it corresponds exactly to the de Rham complex  $\Omega_{X/k}^\bullet$ .

First, we record some properties of the functor  $X \mapsto X^{dR,+}$ .

**Lemma 2.11.** *The functor  $X \mapsto X^{dR,+}$  from  $k$ -schemes to stacks over  $\mathbb{A}^1/\mathbb{G}_m$  satisfies the following properties:*

- (a) *The functor commutes with products.*
- (b) *If  $f : U \rightarrow X$  is étale then the diagram*

$$(2.1) \quad \begin{array}{ccc} U \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & U^{dR,+} \\ \downarrow & & \downarrow \\ X \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & X^{dR,+} \end{array}$$

*is cartesian. Moreover, the vertical functors are étale and if  $f$  is open then the vertical functors are open.*

- (c) *If  $X$  is a colimit of a finite diagram  $U^\bullet$  of affine open subschemes of  $X$ , then  $U^{\bullet,dR,+}$  forms a finite subfunctors of affine open subfunctors of  $X^{dR,+}$  with colimit  $X^{dR,+}$ .*

*Proof.* (a) is by definition. To check (b), we need to prove, for any  $\mathrm{Spec}(R) \rightarrow \mathbb{A}^1/\mathbb{G}_m$ , i.e., a pair  $L \in \mathrm{Pic}(R)$  and a homomorphism  $L \rightarrow R$ , there is an isomorphism

$$X(R) \times_{X(\mathrm{Cone}[\mathrm{Nil}(R) \otimes_R L \rightarrow R])} U(\mathrm{Cone}[\mathrm{Nil}(R) \otimes_R L \rightarrow R]) \simeq U(R).$$

This is equivalent to unique infinitesimal lifting for the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{Cone}[\mathrm{Nil}(R) \otimes_R L \rightarrow R]) & \longrightarrow & U \\ \downarrow & \dashrightarrow & \downarrow f \\ \mathrm{Spec}(R) & \longrightarrow & X. \end{array}$$

For (c), note that since  $U^{\bullet,dR,+}$  are open subfunctors by (b), we have an inclusion

$$\mathrm{colim} U^{\bullet,dR,+} \hookrightarrow X^{dR,+}.$$

We hope to check this is surjective. There is a diagram (2.1) gives

$$\begin{array}{ccc} \mathrm{colim} U^\bullet \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & \mathrm{colim} U^{\bullet,dR,+} \\ \simeq \downarrow & & \downarrow \\ X \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & X^{dR,+} \end{array}$$

By Remark 2.9, we know the horizontal maps are surjective, so the right vertical map must also be surjective, as desired.  $\square$

Now, we can finally:

*Proof of Theorem 2.10.* We proceed in three steps:

- (1) We claim  $\mathcal{H}_{dR,+}(X) \in \mathcal{D}(\mathbb{A}^1/\mathbb{G}_m, \mathcal{O})$  is quasi-coherent and  $t$ -complete, and moreover its restriction to  $B\mathbb{G}_m$  agrees with the natural pushforward map along  $X^{Hodge} \rightarrow B\mathbb{G}_m$ . By Lemma 2.11(c) it suffices to check when there is an étale map  $f: X \rightarrow \mathbb{A}^n$ . By (b) there is a cartesian square

$$\begin{array}{ccc} X \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & X^{dR,+} \\ \downarrow & & \downarrow \\ \mathbb{A}^n \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & (\mathbb{A}^n)^{dR,+}. \end{array}$$

The bottom map is a  $G = \mathbf{V}(\widehat{\mathcal{O}(-1)})^n$ -torsor. Indeed, by (a) it suffices to consider  $n = 1$ , in which case it follows by definition. Now the top horizontal map must be a  $G$ -torsor as well, so we have an isomorphism

$$X^{dR,+} \simeq (X \times \mathbb{A}^1/\mathbb{G}_m)/G.$$

Now by Remark 1.21 the pushforward  $\mathcal{H}_{dR,+}(X)$  is quasi-coherent and is compatible with base change. Moreover,  $t$ -completeness follows from transporting completeness along the equivalence  $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathcal{D}_{fil}(k)$ , as in Theorem 1.9(4).

- (2) By derived deformation theory, there is an isomorphism

$$X^{Hodge} \simeq B\mathbf{V}(\widehat{T_{X/k}(-1)}),$$

where  $T_{X/k}(-1) = pr_1^*T_{X/k} \otimes pr_2^*\mathcal{O}_{B\mathbb{G}_m}(-1)$ . Indeed, given:

- a finite type  $k$ -scheme  $X$ ;
  - an animated  $k$ -algebra  $R$ ;
  - a map  $\eta: \mathrm{Spec}(R) \rightarrow X$  of derived  $k$ -schemes; and
  - a square-zero extension  $R' \rightarrow R$  in animated  $k$ -algebras by  $N \in \mathcal{D}^{\leq 0}(R)$ ,
- the fiber of the map  $X(R') \rightarrow X(R)$  over  $\eta \in X(R)$  is a torsor for

$$\mathrm{Der}(\mathcal{O}_X, \eta_*N) \simeq \mathrm{Map}_R(\eta^*L_{X/k}, N).$$

When  $X$  is smooth and  $N = L[1] \in \mathcal{D}^{\leq -1}(R)$ , we have

$$\mathrm{Map}_R(\eta^*L_{X/k}, N) \simeq B(\eta^*T_{X/k} \otimes_R L).$$

When furthermore  $R' \rightarrow R$  is split, the fiber of  $X(R') \rightarrow X(R)$  is a split torsor, canonically identified with  $B(\eta^*T_{X/k} \otimes_R L)$ .

Let us apply this to the square-zero extension

$$R \oplus \mathrm{Nil}(R)(-1)[1] \rightarrow R$$

by  $\mathrm{Nil}(R)(-1)[1]$ , so  $X^{Hodge} \rightarrow X$  is a split torsor with fibers

$$B(\eta^*T_{X/k} \otimes_R \mathrm{Nil}(R) \otimes_R L),$$

i.e.,

$$X^{Hodge} \simeq B\mathbf{V}(\widehat{T_{X/k}(-1)}).$$

Now, Proposition 1.19, gives an equivalence

$$\mathcal{D}_{qc}(X^{Hodge}) \simeq \mathcal{D}_{qc}(\mathbf{V}(\Omega_{X/k}(1))),$$

sending  $\mathcal{O}_{X^{Hodge}}$  to  $\mathcal{O}_{X \times \mathbb{G}_m}$ . To compute the pushforward  $\pi_{X,*} \mathcal{O}_{X^{Hodge}}$ , note that under the equivalence this is equivalent to computing

$$RHom_{\mathbf{V}(\Omega_{X/k}(1))}(\mathcal{O}_{X \times \mathbb{G}_m}, \mathcal{O}_{X \times \mathbb{G}_m}).$$

But the Koszul resolution provides a quasi-isomorphism between  $\mathcal{O}_{X \times \mathbb{G}_m}$  and

$$[\cdots \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))} \otimes_{\mathcal{O}_{X \times B\mathbb{G}_m}} \wedge^2 T_{X/k}(-2) \rightarrow \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))} \otimes_{\mathcal{O}_{X \times B\mathbb{G}_m}} T_{X/k}(-1) \rightarrow \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))}],$$

which provides an isomorphism

$$\begin{aligned} RHom_{\mathbf{V}(\Omega_{X/k}(1))}(\mathcal{O}_{X \times \mathbb{G}_m}, \mathcal{O}_{X \times \mathbb{G}_m}) &\simeq RHom_{\mathbf{V}(\Omega_{X/k}(1))}([\cdots \rightarrow \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))} \otimes T_{X/k}(-1) \rightarrow \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))}], \mathcal{O}_{X \times \mathbb{G}_m}) \\ &\simeq RHom_{X \times \mathbb{G}_m}(\cdots \xrightarrow{0} T_{X/k}(-1) \xrightarrow{0} \mathcal{O}_{X \times \mathbb{G}_m}, \mathcal{O}_{X \times \mathbb{G}_m}) \\ &\simeq \bigoplus_i R\Gamma(X, \Omega_{X/k}^i[-i])(i). \end{aligned}$$

- (3) We have a presheaf  $\mathcal{F}: U \mapsto \mathcal{H}_{dR,+}(U)$  on  $X$  valued in  $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathcal{D}_{fil}(k)$ , which is a sheaf by (c). Moreover, by the first two parts, the sheaf  $\mathcal{F}$  lies in the heart of the Beilinson  $t$ -structure. Thus, it is represented by a chain complex, which by (2) must be of the form

$$\mathcal{O}_X \xrightarrow{\delta} \Omega_{X/k}^1 \xrightarrow{\delta} \Omega_{X/k}^2 \xrightarrow{\delta} \cdots,$$

for some differentials  $\delta$ , equipped with the stupid filtration. To prove the theorem, we need only check that  $\delta$  are the de Rham differentials. By Lemma 2.11, it suffices to show the case  $X = \mathbb{A}^1$ . In this case,  $X^{dR,+} = \mathbb{G}_a^{dR,+}$  and by Remark 1.21 the cohomology  $\mathcal{H}_{dR,+}(X)$  is computed by the graded  $k[t]$ -complex

$$k[t, x] \xrightarrow{t \frac{d}{dx}} k[t, x](1),$$

since the differential is  $\frac{d}{dx}$  on the non-filtered objects. Thus translating to filtered objects, we see that  $\delta = \frac{d}{dx}$ .  $\square$

The stack  $X^{dR,+}$  not only geometrizes de Rham cohomology, but it also geometrizes the category of vector bundles with flat connections.

**Remark 2.12.** By pullback along the map  $X \times \mathbb{A}^1/\mathbb{G}_m \rightarrow X^{dR,+}$  from Remark 2.9, the category  $\text{Vect}(X^{dR,+})$  of vector bundles on  $X^{dR,+}$  is identified with the category of triples  $(E, \nabla, F^*)$  where:

- $E$  is a vector bundle on  $X$ ;
- $\nabla: E \rightarrow \Omega_{X/k}^1 \otimes_{\mathcal{O}_X} E$  is a flat connection; and
- $F^*$  if a finite filtration of  $E$  by submodules satisfying Griffiths transversality:

$$\nabla(F^i) \subset \Omega_{X/k}^1 \otimes_{\mathcal{O}_X} F^{i-1}.$$

Similarly, the pullback along  $X \times B\mathbb{G}_m \rightarrow X^{Hodge}$  identifies  $\text{Vect}(X^{Hodge})$  with the category of graded Higgs bundles, i.e., graded vector bundles  $M = \bigoplus_i M_i$  together with a Higgs field  $\Theta: M \rightarrow \Omega_{X/k}^1 \otimes_{\mathcal{O}_X} M$  (i.e., such that  $\Theta \wedge \Theta = 0$ ) taking  $M_i$  to  $\Omega_{X/k}^1 \otimes_{\mathcal{O}_X} M_{i+1}$ . Under this description,  $\text{Vect}(X^{dR,+}) \rightarrow \text{Vect}(X^{Hodge})$  simply takes associated graded.

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