ALGEBRAIC DE RHAM COHOMOLOGY VIA STACKS

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Abstract. We cover Chapter 2 of Bhatt's notes [Bha22] on Primstic F-gauges.

1. Linear algebra via stacks

Let R be a commutative ring. We hope to express the category of graded R-modules and filtered R-modules using the language of stacks.

1.1. **Graded R-modules.** The derived category of graded R-modules is defined as $\mathcal{D}_{gr}(R) := \mathrm{Fun}(\mathbb{Z}, \mathcal{D}(R))$, where $\mathbb{Z}$ is considered a discrete category. Concretely, the objects of $\mathcal{D}_{gr}(R)$ is just a collection of objects $F(i) \in \mathcal{D}(R)$ indexed by integers $i \in \mathbb{Z}$. This is a symmetric monoidal category, with tensor product defined by:

$$(F \otimes G)(n) := \bigoplus_{i+j=n} F(i) \otimes G(j).$$

Now we can re-write $\mathcal{D}_{gr}(R)$ using the language of stacks as follows. First, recall that $B\mathbb{G}_m$ is the stack classifying line bundles on R-schemes, so it carries a tautological line bundle $\mathcal{O}(1)$.

Proposition 1.2. *There is an equivalence of monoidal categories*

$$\mathcal{D}_{gr}(R) \simeq \mathcal{D}_{qc}(B\mathbb{G}_m)$$

defined by

$$F \mapsto \bigoplus_{i\in\mathbb{Z}} F(i) \otimes_R \mathcal{O}(-i),$$

with inverse defined by

$$M \mapsto \Big(i \mapsto R\Gamma\big(B\mathbb{G}_m, M(i)\big)\Big).$$

Moreover, it fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{gr}(R) & \xrightarrow{\simeq} & \mathcal{D}_{qc}(B\mathbb{G}_m) \\ & {\scriptstyle Forg}\searrow \quad \swarrow{\scriptstyle \pi^*} & \\ & \mathcal{D}(R), & \end{array}$$

where the functor $Forg$ forgets the grading (i.e., is $M \mapsto \bigoplus_i M(i)$) and π is the map $\mathrm{Spec}(R) \to B\mathbb{G}_m$.

1.3. **Filtered R-modules.** Next, we hope to provide a similar description for the category of filtered R-modules. In the non-derived setting, filtered R-modules were defined as follows:

Definition 1.4. A *filtered R-module* is a R-module F together with a sub-modules $\mathrm{Fil}^i F$ indexed by $i \in \mathbb{Z}$ such that $\mathrm{Fil}^{i+1} F \subset \mathrm{Fil}^i F$. A filtered R-module is *exhaustive* if

$$F = \bigcup_{i\in\mathbb{Z}} \mathrm{Fil}^i F.$$

Filtered R-modules can be visualized as a chain:

$$\cdots \subset \operatorname{Fil}^{i+1} F \subset \operatorname{Fil}^{i} F \subset \operatorname{Fil}^{i-1} F \subset \cdots .$$

We want to define the derived category of filtered R-modules $\mathcal{D}_{fil}(R)$. In the derived category, it *does not* make sense to talk about sub-modules, so we instead replace the inclusions $\operatorname{Fil}^{i+1} M \hookrightarrow \operatorname{Fil}^{i} M$ by arbitrary maps. this gives the following:

Definition 1.5. The *derived category of filtered R-modules* is

$$\mathcal{D}_{fil}(R) := \operatorname{Fun}\left(\mathbb{Z}_{\geq}^{op}, \mathcal{D}(R)\right),$$

where $\mathbb{Z}_{\geq}$ is the usual poset of integers considered as a category. We denote the value of a functor F at $i \in \mathbb{Z}$ as $\operatorname{Fil}^i F$.

Now, a derived filtered R-module can be visualized as a chain:

$$\cdots \to \operatorname{Fil}^{i+1} F \to \operatorname{Fil}^{i} F \to \operatorname{Fil}^{i-1} F \to \cdots .$$

We give two sources of filtered R-modules:

Example 1.6 (canonical filtration)**.** There is a fully faithful embedding

$$\mathcal{D}(R) \hookrightarrow \mathcal{D}_{fil}(R),$$

which associates to $K \in \mathcal{D}(R)$ the filtered R-module $\widetilde{K} \in \mathcal{D}_{fil}(R)$ given by

$$\operatorname{Fil}^i \widetilde{K} := \tau^{\leq -i} K.$$

Here $\operatorname{gr}^i \widetilde{K} = (H^{-i}K)[i]$. In fact, the essential image is exactly those $F \in \mathcal{D}_{fil}(R)$ such that $\operatorname{gr}^i F$ is concentrated in cohomological degree $-i$ and is complete in the sense defined below.

Example 1.7 (stupid filtration)**.** There is a fully faithful functor

$$\operatorname{Ch}(R) \hookrightarrow \mathcal{D}_{fil}(R)$$

sending a chain complex $K^\bullet$ of R-modules to

$$\operatorname{Fil}^i K^\bullet = K^{\geq i}.$$

Here $\operatorname{gr}^i K^\bullet = K^i[-i]$. In fact, the essential image is exactly those $F \in \mathcal{D}_{fil}(R)$ such that $\operatorname{gr}^i F$ is concentrated in cohomological degree i.

By analogy to non-derived filtered R-modules, we can define the following notions:

- The *underlying object* is $\underline{F} := \operatorname{colim}_i \operatorname{Fil}^i F$. For an non-derived exhaustive filtered R-module this is the usual notion of an underlying R-module.
- There is a symmetric monoidal structure on $\mathcal{D}_{fil}(R)$ defined by

$$\operatorname{Fil}^n(F \otimes G) = \operatorname{colim}_{i+j \geq n} \operatorname{Fil}^i F \otimes \operatorname{Fil}^j G.$$

- We let $\operatorname{gr}^i_{\operatorname{Fil}} F := \operatorname{Cone}(\operatorname{Fil}^{i+1} F \to \operatorname{Fil}^i F)$. The construction

$$F \mapsto \operatorname{gr}^*_{\operatorname{Fil}} F = \bigoplus_i \operatorname{gr}^i_{\operatorname{Fil}} F$$

  gives an exact colimit-preserving symmetric monoidal functor

$$\mathcal{D}_{fil}(R) \to \mathcal{D}_{gr}(R).$$

  This is the derived analog of the classical construction

$$\operatorname{gr}^*_{\operatorname{Fil}} F = \bigoplus_i \operatorname{Fil}^i F / \operatorname{Fil}^{i+1} F.$$

- When F is a non-derived filtered R-module, the filtration is *complete* when
$$\bigcap_{i\in\mathbb{Z}} \operatorname{Fil}^i F = 0.$$
Analogously, for $F \in \mathcal{D}_{fil}(R)$, we say F is *complete* when
$$\lim_i \operatorname{Fil}^i F = 0.$$
Let $\widehat{\mathcal{D}}_{fil}(R) \subset \mathcal{D}_{fil}(R)$ be the full subcategory of complete filtered R-modules. The inclusion has a left-adjoint given by
$$\widehat{F} := \operatorname{Cone}\big(\operatorname{Const}(\lim_i \operatorname{Fil}^i F) \to F\big),$$
where for $X \in \mathcal{D}(R)$, we let $\operatorname{Const}(X)$ denote the constant functor $\operatorname{Fil}^i \operatorname{Const}(X) = X$.
- Given $F \in \mathcal{D}_{fil}(R)$, we can define $F\{n\}$ to be shift by n, i.e.,
$$\operatorname{Fil}^i(F\{n\}) := \operatorname{Fil}^{i+n} F.$$
There is a natural map $F\{1\} \to F$, given by the map
$$\operatorname{Fil}^i(F\{1\}) = \operatorname{Fil}^{i+1} F \to \operatorname{Fil}^i F,$$
such that
$$\operatorname{Fil}^i(F/F\{1\}) = \operatorname{gr}^i F.$$
- There are two natural t-structures on $\mathcal{D}_{fil}(R)$:
  - The standard t-structure: $F \in \mathcal{D}_{fil}(R)$ is connective (resp., co-connective) when $\operatorname{Fil}^i F$ is connective (resp., co-connective) for each i
  - The Beilinson t-structure: $F \in \mathcal{D}_{fil}(R)$ is connective (resp., co-connective) when $\operatorname{gr}^i F \in \mathcal{D}^{\leq i}$ (resp., $\operatorname{gr}^i F \in \mathcal{D}^{\geq i}$) for each $i \in \mathbb{Z}$. By Example 1.7, the stupid filtration gives an equivalence between $\operatorname{Ch}(R)$ and the heart of the Beilinson t-structure.

We ultimately hope to prove an equivalence between $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$ and $\mathcal{D}_{fil}(R)$. Here, we let $\mathbb{A}^1 = \operatorname{Spec} R[t]$ and let the $\mathbb{G}_m$-action give t degree 1. Thus, there is an equivalence $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathcal{D}_{gr}(R[t])$, where $\mathcal{D}_{gr}(R[t])$ is the derived category of graded $R[t]$-modules. First, let us recall what the stack $\mathbb{A}^1/\mathbb{G}_m$ classifies:

Remark 1.8. Given a scheme T, the groupoid $\mathbb{A}^1/\mathbb{G}_m(T)$ classifies $\mathbb{G}_m$-torsors $T' \to T$ together with a $\mathbb{G}_m$-equivariant map $T' \to \mathbb{A}^1$. A $\mathbb{G}_m$-torsor must be of the form
$$T' = \operatorname{Spec}(\bigoplus_{i\in\mathbb{Z}} \mathcal{L}^{-i})$$
for a line bundle $\mathcal{L}$ on T, and a map $T' \to \mathbb{A}^1$ is equivalent to a $\mathcal{O}_T$-linear map $\mathcal{O}_T \to \mathcal{L}^{-1}$. Thus
$$\mathbb{A}^1/\mathbb{G}_m(T) \simeq \{\text{a line bundle } \mathcal{L} \text{ on } T\text{, with a } \mathcal{O}_T\text{-linear map } \mathcal{L} \to \mathcal{O}_T\}.$$
In particular, there is a universal pair $t\colon \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}(-1) \to \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}$ over $\mathbb{A}^1/\mathbb{G}_m$. As graded $R[t]$-modules, this is the inclusion
$$tR[t] \subset R[t]$$
of graded $R[t]$-modules. The vanishing locus of t is the Cartier divisor $B\mathbb{G}_m \subset \mathbb{A}^1/\mathbb{G}_m$. Moreover, $\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}(t)|_{B\mathbb{G}_m} \simeq \mathcal{O}_{B\mathbb{G}_m}(-1)$, since the graded $R[t]$-module $tR[t]/t^2R[t]$ is a copy of R in degree 1.

Now, the main theorem is:

Theorem 1.9. *There is an equivalence of symmetric monoidal categories*

$$\mathrm{Rees}\colon \mathcal{D}_{fil}(R) \simeq \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$$

defined by sending $F \in \mathcal{D}_{fil}(R)$ to the graded $R[t]$-module

$$\mathrm{Rees}(F) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i F \cdot t^{-i}.$$

It has the following properties:

(1) Rees *is t-exact with the standard t-structures.*
(2) *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{D}_{fil}(R) & \xrightarrow[\simeq]{Rees} & \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m) \\ & {\scriptstyle Forg}\searrow \quad \swarrow{\scriptstyle j^*} & \\ & \mathcal{D}(R), & \end{array}$$

where $j\colon \mathrm{Spec}(R) = \mathbb{G}_m/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$ is the standard open immersion and $Forg$ forgets the filtration (i.e., takes the underlying module.)
(3) *Restriction to the Cartier divisor $i\colon B\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$ corresponds to passing to the associated graded, up to a change of sign. More precisely, for $F \in \mathcal{D}_{fil}(R)$ and $i \in \mathbb{Z}$,*

$$i^* \mathrm{Rees}(F) = \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i_{\mathrm{Fil}} F \otimes \mathcal{O}(i),$$

or, equivalently,

$$\mathrm{gr}^i_{\mathrm{Fil}} F \simeq R\Gamma(B\mathbb{G}_m, i^* \mathrm{Rees}(F)(-i)).$$

(4) *$F \in \mathcal{D}_{fil}(R)$ is complete as a filtered R-module if and only if* $\mathrm{Rees}(F) \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$ *is* derived t-complete. *Here, $M \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$ is* derived t-complete *when the derived limit of the diagram*

$$\cdots \xrightarrow{t} M \xrightarrow{t} M \xrightarrow{t} M$$

is zero.
(5) *For any $F \in \mathcal{D}_{fil}(R)$ there is an isomorphism* $\mathrm{Rees}(F\{n\}) \simeq \mathrm{Rees}(F) \otimes \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}(-n)$.

Proof Sketch. Given an object $M \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$, we can consider the filtered R-module given by taking $R\Gamma(\mathbb{A}^1/\mathbb{G}_m, -)$ of the diagram

$$\cdots \xrightarrow{t} M \otimes_{\mathcal{O}} \mathcal{O}(i-1) \xrightarrow{t} M \otimes_{\mathcal{O}} \mathcal{O}(i) \xrightarrow{t} M \otimes_{\mathcal{O}} \mathcal{O}(i+1) \xrightarrow{t} \cdots.$$

In the language of graded $R[t]$-modules, given $M = \bigoplus_i M(i) \in \mathcal{D}_{gr}(R[t])$, we can take

$$\mathrm{Fil}^i(\mathrm{Rees}^{-1} M) = M(-i)$$

with transition maps $t\colon M(-i) \to M(-i+1)$. □

Remark 1.10. Any perfect complex $M \in \mathrm{Perf}(\mathbb{A}^1/\mathbb{G}_m)$ is derived t-complete. Indeed, it suffices to check this when $M = \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}$, which is essentially the fact that $k[t] \simeq \lim k[t]/t^n$ as graded vector spaces.

Remark 1.11 (Vector bundles on $\mathbb{A}^1/\mathbb{G}_m$)**.** Under the Rees equivalence, the category $\mathrm{Vect}(\mathbb{A}^1/\mathbb{G}_m)$ of vector bundles on $\mathbb{A}^1/\mathbb{G}_m$ is identified with the category of pairs (M, F^*) where M is a finite projective R-module and F^* is a finite exhaustive filtration on M (in the non-derived sense) such that $\mathrm{gr}^i_F M$ is finite projective for all i.

Remark 1.12 (Canonical filtrations)**.** The composition of functors from Example 1.6 and the theorem gives
$$\mathcal{D}(R) \hookrightarrow \mathcal{D}_{fil}(R) \simeq \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m).$$
The essential image consists of objects $M \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$ which are complete and such that $\mathcal{H}^i(M)(i)$ is constant, i.e., pulled back from $\mathrm{Spec}(R)$, for all i.

Definition 1.13. A *filtered stack* is a stack $\mathfrak{X}$ together with a morphism $f\colon \mathfrak{X} \to \mathbb{A}^1/\mathbb{G}_m$.

Remark 1.14. A filtered stack can be viewed as a filtratio on the stack
$$\underline{\mathfrak{X}} := f^{-1}(\mathbb{G}_m/\mathbb{G}_m)$$
with associated graded
$$\mathrm{Gr}(\mathfrak{X}) := f^{-1}(B\mathbb{G}_m).$$
Assuming f_* preserves quasi-coherence (e.g., when f is representable qcqs), for any $M \in \mathcal{D}_{qc}(\mathfrak{X})$, the pushforward
$$f_*M \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathcal{D}_{fil}(R)$$
is a filtration on $R\Gamma(\underline{\mathfrak{X}}, M)$ with associated graded $R\Gamma(\mathrm{Gr}(\mathfrak{X}), M)$.

1.15. **Endomorphisms and $B\widehat{\mathbb{G}}_a$.**

Definition 1.16. Let $\widehat{\mathbb{G}}_a \subset \mathbb{G}_a$ be the formal completion at 0; the functor of points is $\widehat{\mathbb{G}}_a(S) = \mathrm{Nil}(S)$ for any R-algebra S.

Then we have the proposition:

Proposition 1.17. *Let R be a commutative $\mathbb{Q}$-algebra. There is an equivalence of symmetric monoidal categories*
$$\Phi\colon \mathcal{D}_{qc}(B\widehat{\mathbb{G}}_a) \simeq \mathcal{D}(R[t]),$$
where $\mathcal{D}(R[t])$ is a symmetric monoidal category under convolution, i.e., for $M, N \in \mathcal{D}(R[t])$ the convolution $M \star N$ has underlying R-module $M \otimes_R N$ and t acts via $t_M \otimes 1_N + 1_M \otimes t_N$. The functor Φ has properties:

(1) There is a commutative diagram
$$\begin{array}{ccc} \mathcal{D}_{qc}(B\widehat{\mathbb{G}}_a) & \xrightarrow[\simeq]{\Phi} & \mathcal{D}(R[t]) \\ & {\scriptstyle \pi^*}\searrow \quad \swarrow{\scriptstyle Forg} & \\ & \mathcal{D}(R), & \end{array}$$
where $\pi\colon \mathrm{Spec}(R) \to B\widehat{\mathbb{G}}_a$ is the standard map and $Forg$ forgets the action of t.

(2) Φ sends $\mathcal{O}_{B\widehat{\mathbb{G}}_a}$ to $R[t]/(t) \simeq R$. Thus, for $M \in \mathcal{D}_{qc}(B\widehat{\mathbb{G}}_a)$, there is a natural isomorphism
$$R\Gamma(B\widehat{\mathbb{G}}_a, M) \simeq \mathrm{RHom}_{k[t]}(k, \Phi(M)) \simeq \mathrm{Fib}(\Phi(M) \xrightarrow{t} \Phi(M)).$$
In particular, $R\Gamma(B\widehat{\mathbb{G}}_a, -)$ has cohomological dimension 1.

Proof. Write X for the coordinate on $\widehat{\mathbb{G}}_a$ that is $\mathbb{G}_m$-equivariantly dual to t. Then for $M \in \mathcal{D}(R[t])$, the corresponding quasi-coherent sheaf on $B\widehat{\mathbb{G}}_a$ is given by the co-action map
$$c\colon M \to M[\![X]\!]$$
$$m \mapsto \exp(tX)m := \sum_{i \geq 0} t^i(m)\frac{X^i}{i!}.$$
Moreover, the $R[t]$-module structure can be recovered as the coefficient of X. □

Example 1.18. The inclusion $\widehat{\mathbb{G}}_a \subset \mathbb{G}_a$ gives a $k[t]$-module structure on $\mathcal{O}(\mathbb{G}_a) = k[X]$, which is simply $t = \frac{d}{dX}$.

In other words, $\widehat{\mathbb{G}}_a$-representations are equivalent to modules with an endomorphism.

In fact, Proposition 1.17 can be upgraded to work in families. Given a finite projective R-module E, consider the associated vector bundle $\mathbf{V}(E)$. We can analogously define $\widehat{\mathbf{V}(E)}$. Then we have:

Proposition 1.19. *Let R be a commutative $\mathbb{Q}$-algebra and let E be a finite projective R-module. Then there is a natural equivalence of symmetric monoidal categories*

$$\mathcal{D}_{qc}(B\widehat{\mathbf{V}(E)}) \simeq \mathcal{D}_{qc}(\mathbf{V}(E^\vee))$$

where $\mathcal{D}_{qc}(\mathbf{V}(E^\vee))$ is given the convolution product. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{qc}(B\widehat{\mathbf{V}(E)}) & \xrightarrow[\simeq]{\Phi} & \mathcal{D}_{qc}(\mathbf{V}(E^\vee)) \\ & {\scriptstyle \pi^*}\searrow \quad \swarrow{\scriptstyle s_*} & \\ & \mathcal{D}(R), & \end{array}$$

where as usual $\pi\colon \operatorname{Spec}(R) \to B\widehat{\mathbf{V}(E)}$ is the tautological map, and s is the structure map $\mathbf{V}(E^\vee) \to \operatorname{Spec}(R)$.

Remark 1.20. We can use the Proposition to compute the cohomology of $\widehat{\mathbf{V}(E)}$-representations. Recall that an object $M \in \mathcal{D}_{qc}(B\widehat{\mathbf{V}(E)})$ can be regarded as a $\widehat{\mathbf{V}(E)}$-representation on $\pi^*M \in \mathcal{D}(R)$. By the proposition, π^*M carries a natural action of $S = \operatorname{Sym}^*_R(E)$, and

$$R\Gamma(B\widehat{\mathbf{V}(E)}, M) := \operatorname{RHom}_{B\widehat{\mathbf{V}(E)}}(\mathcal{O}, M) \simeq \operatorname{RHom}_S(R, \pi^*M).$$

The derived Hom can be computed using the Koszul resolution of R.

We need a relative version of the Remark 1.20:

Remark 1.21. Suppose we have a qcqs morphism $f\colon Y \to Z$ of characteristic 0 schemes, a line bundle $\mathcal{L}$ on Z, and a Z-linear action of $G = \widehat{\mathbf{V}(\mathcal{L})}$ on Y. Then we have a recipe to compute pushforwards along $f_G\colon Y/G \to Z$. Consider the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\ Y/G & \xrightarrow{\widetilde{f}} & BG. \end{array}$$

The horizontal map is qcqs, so the pushforward along the map preserves quasi-coherence. Moreover, given $M \in \mathcal{D}_{qc}(Y/G)$, flat base change shows

$$\pi_Z^* R\widetilde{f}_* M \simeq Rf_*\pi_Y^* M.$$

Pushing forward $R\widetilde{f}_*M$ along the structure map $g\colon BG \to Z$ and using Remark 1.20, we learn that $Rf_{G,*}M \simeq Rg_*R\widetilde{f}^*M$ sits in a fiber sequence

$$Rf_{G,*}M \to Rf_*\pi_Y^*M \to Rf_*\pi_Y^*M \otimes \mathcal{L}^{-1}.$$

Thus, $Rf_{G,*}M$ is quasi-coherent.

Analogously, there are equivalences:

$$\mathcal{D}_{qc}(B\mathbb{G}_a) \simeq \mathcal{D}_{qc}(\widehat{\mathbb{G}}_a) \tag{1.1}$$

$$\mathcal{D}_{qc}(B\mathbf{V}(E)) \simeq \mathcal{D}_{qc}(\widehat{\mathbf{V}(E^\vee)}), \tag{1.2}$$

swapping the role of $\mathbb{G}_a$ and $\widehat{\mathbb{G}}_a$.

We will use the following variant of Serre vanishing:

Lemma 1.22. *Let R be any commutative ring. Then*

$$R\Gamma_{et}(\mathrm{Spec}(R), \widehat{\mathbb{G}}_a) \simeq \mathrm{Nil}(R)[0].$$

Proof. It suffices to check that for any étale cover $R \to S$ with Cech nerve $R \to S^\bullet$ we have

$$\mathrm{Nil}(R) \simeq \lim \mathrm{Nil}(S^\bullet).$$

Since $R \to S$ is étale, $\mathrm{Nil}(R) \otimes_R S^\bullet \simeq \mathrm{Nil}(S^\bullet)$ so

$$\lim \mathrm{Nil}(S^\bullet) \simeq \lim \mathrm{Nil}(R) \otimes_R S^\bullet \simeq \mathrm{Nil}(R)$$

by fpqc descent for quasi-coherent sheaves. □

2. de Rham cohomology in characteristic 0 via stacks

In this section, we work over a ground field k of characteristic 0.

Definition 2.1. The scheme $\mathbb{G}_a$ is naturally a ring scheme, and the subfunctor $\widehat{\mathbb{G}}_a \subset \mathbb{G}_a$ is an ideal group scheme. Thus, the quotient sheaf $\mathbb{G}_a^{dR} := \mathbb{G}_a/\widehat{\mathbb{G}}_a$ is a sheaf of rings, and for any ring R,

$$\mathbb{G}_a^{dR}(R) = R_{red}.$$

Remark 2.2. Of course, $\mathbb{G}_a(R)/\widehat{\mathbb{G}}_a(R) = R/\mathrm{Nil}(R) = R_{red}$. The fact that even upon sheafification we have $\mathbb{G}_a^{dR}(R) = R_{red}$ is due to Lemma 1.22.

Now, using the ring stack $\mathbb{G}_a^{dR}$, we can define the de Rham space:

Definition 2.3. For any k-scheme X, let the *de Rham space* X^{dR} be the functor on finite-type k-algebras given by

$$X^{dR}(R) := X(\mathbb{G}_a^{dR}(R)) = X(R_{red}).$$

Remark 2.4. In general, there is a natural map $X \to X^{dR}$ induced by the quotient map $\mathbb{G}_a \to \mathbb{G}_a^{dR}$. When X is smooth, this map $X \to X^{dR}$ is a surjection of étale sheaves, by the infinitesimal lifting property of smoothness. For any k-algebra T, we claim $X(T) \to X(T^{red})$ is surjective. But the infinitesimal lifting property states that in the diagram

$$\begin{array}{ccc} \mathrm{Spec}(T^{red}) & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \mathrm{Spec}(T) & \longrightarrow & \mathrm{Spec}(k), \end{array}$$

there exists a lifting $\mathrm{Spec}(T) \to X$, which is exactly what we want.

Now our goal is to show:

Theorem 2.5 (de Rham cohomology via X^{dR})**.** *For a smooth k-scheme X, there is a natural identification*

$$R\Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) \simeq R\Gamma(X, \Omega^\bullet_{X/k}).$$

Under this isomorphism, pulling back along $X \to X^{dR}$ corresponds to the projection

$$\mathrm{gr}^0_H R\Gamma(X, \Omega^\bullet_{X/k}) \simeq R\Gamma(X, \mathcal{O}_X)$$

given by the Hodge filtration.

To prove the theorem, we construct a filtration on $R\Gamma(X^{dR}, \mathcal{O}_{X^{dR}})$, and, in fact, on X^{dR}. Recall that filtering X^{dR} means finding a stack $\mathfrak{X} \to \mathbb{A}^1/\mathbb{G}_m$ such that $\mathfrak{X}|_{\mathbb{G}_m/\mathbb{G}_m} \simeq X^{dR}$.

Definition 2.6. Consider the universal effective Cartier divisor $t\colon \mathcal{O}(-1) \to \mathcal{O}$ on the stack $\mathbb{A}^1/\mathbb{G}_m$. Passing to the associated vector bundle schemes, we have a morphism

$$d\colon \mathbf{V}(\widehat{\mathcal{O}(-1)}) \xrightarrow{t} \mathbf{V}(\mathcal{O}) = \mathbb{G}_a.$$

over $\mathbb{A}^1/\mathbb{G}_m$. Now, the stack quotient

$$\mathbb{G}_a^{dR,+} = \mathrm{Cone}(\mathbf{V}(\widehat{\mathcal{O}(-1)}) \to \mathbb{G}_a)$$

becomes a 1-truncated animated $\mathbb{G}_a$-algebra over $\mathbb{A}^1/\mathbb{G}_m$. In other words, if a map $\mathrm{Spec}(R) \to \mathbb{A}^1/\mathbb{G}_m$ is given by $(L \in \mathrm{Pic}(R), L \to R)$, then

$$\mathbb{G}_a^{dR,+}(\mathrm{Spec}(R) \to \mathbb{A}^1/\mathbb{G}_m) = \mathrm{Cone}(\mathrm{Nil}(R) \otimes_R L \to R).$$

Remark 2.7. There are isomorphisms

$$\mathbb{G}_a^{dR,+}|_{\mathbb{G}_m/\mathbb{G}_m} \simeq \mathbb{G}_a^{dR}$$
$$\mathbb{G}_a^{dR,+}|_{B\mathbb{G}_m} \simeq \mathbb{G}_a^{Hodge},$$

where

$$\mathbb{G}_a^{Hodge} := \mathbb{G}_a \oplus \mathbf{V}(\widehat{\mathcal{O}(-1)})[1],$$

i.e.,

$$\mathbb{G}_a^{Hodge}(\mathrm{Spec}(R) \to B\mathbb{G}_m) = R \oplus \mathrm{Nil}(R) \otimes_R L[1].$$

Indeed, over $\mathbb{G}_m/\mathbb{G}_m$ the map $L \to R$ is an isomorphism, so

$$\mathrm{Cone}(\mathrm{Nil}(R) \otimes_R L \to R) \simeq \mathrm{Cone}(\mathrm{Nil}(R) \to R) = R_{red}$$

and over $\mathbb{A}^1/\mathbb{G}_m$, the map $L \to R$ is zero so

$$\mathrm{Cone}(\mathrm{Nil}(R) \otimes_R L \to R) \simeq R \oplus \mathrm{Nil}(R) \otimes_R L[1]$$

Now, we can define the filtered de Rham stack:

Definition 2.8. For a smooth k-scheme X, the *filtered de Rham space* is the map $X^{dR,+} \to \mathbb{A}^1/\mathbb{G}_m$ whose functor of points is

$$X^{dR,+}(\mathrm{Spec}(R) \to \mathbb{A}^1/\mathbb{G}_m) = X(\mathbb{G}_a^{dR,+}(R)),$$

where the right-hand side is the groupoid of maps $\mathrm{Spec}(\mathbb{G}_a^{dR,+}(R)) \to X$ in derived algebraic geometry. The fiber

$$X^{Hodge} := X^{dR,+} \times_{\mathbb{A}^1/\mathbb{G}_m} B\mathbb{G}_m$$

is called the *Hodge stack* of X, so the functor X^{Hodge} on $B\mathbb{G}_m$-schemes is given by

$$X^{Hodge}(\mathrm{Spec}(R) \to B\mathbb{G}_m) = X(\mathbb{G}_a^{Hodge}(R)).$$

The filtered de Rham stack recovers X^{dR} over $\mathbb{G}_m/\mathbb{G}_m$:

$$X^{dR} \simeq X^{dR,+} \times_{\mathbb{A}^1/\mathbb{G}_m} \mathbb{G}_m/\mathbb{G}_m.$$

Remark 2.9. Generalizing Remark 2.4, the quotient maps $\mathbb{G}_a \to \mathbb{G}_a^{dR}$, $\mathbb{G}_a \to \mathbb{G}_a^{Hodge}$, and $\mathbb{G}_a \to \mathbb{G}_a^{dR,+}$ induce maps

$$X \to X^{dR}$$
$$X \times B\mathbb{G}_m \to X^{Hodge}$$
$$X \times \mathbb{A}^1/\mathbb{G}_m \to X^{dR,+}.$$

When X is smooth, all of these are surjections of étale sheaves.

Now, Theorem 2.5 follows from the stronger theorem:

Theorem 2.10 (Hodge-filtered de Rham cohomology via $X^{dR,+}$)**.** *For X/k a smooth variety, let $\pi_X\colon X^{dR,+} \to \mathbb{A}^1/\mathbb{G}_m$ be the structure map. Then*

$$\mathcal{H}_{dR,+}(X) := R\pi_*\mathcal{O}_{X^{dR,+}}$$

is quasi-coherent and complete, and the corresponding filtered object of $\widehat{\mathcal{D}}_{fil}(k)$ identifies with the Hodge-filtered de Rham cohomology $\mathrm{Fil}^*_H R\Gamma(X, \Omega^\bullet_{X/k})$.

In fact, what we prove is that

$$U \mapsto \mathcal{H}_{dR,+}(U) \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$$

can be regarded as a Zariski sheaf $\mathcal{F}$ on X valued in $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$. The category

$$\mathrm{Shv}(X, \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)) \simeq \mathrm{Shv}(X, \mathcal{D}_{fil}(k))$$

carries a Beilinson t-structure, whose heart is the abelian category of chain complexes of sheaves of k-modules on X. We show that $U \mapsto \mathcal{H}_{dR,+}(U)$ lies in the heart of the t-structure, and it corresponds exactly to the de Rham complex $\Omega^\bullet_{X/k}$.

First, we record some properties of the functor $X \mapsto X^{dR,+}$.

Lemma 2.11. *The functor $X \mapsto X^{dR,+}$ from k-schemes to stacks over $\mathbb{A}^1/\mathbb{G}_m$ satisfies the following properties:*

(a) The functor commutes with products.
(b) If $f\colon U \to X$ is étale then the diagram

$$\begin{array}{ccc} U \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & U^{dR,+} \\ \downarrow & & \downarrow \\ X \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & X^{dR,+} \end{array} \tag{2.1}$$

is cartesian. Moreover, the vertical functors are étale and if f is open then the vertical functors are open.
(c) If X is a colimit of a finite diagram $U^\bullet$ of affine open subschemes of X, then $U^{\bullet,dR,+}$ forms a finite subfunctors of affine open subfunctors of $X^{dR,+}$ with colimit $X^{dR,+}$.

Proof. (a) is by definition. To check (b), we need to prove, for any $\mathrm{Spec}(R) \to \mathbb{A}^1/\mathbb{G}_m$, i.e., a pair $L \in \mathrm{Pic}(R)$ and a homomorphism $L \to R$, there is an isomorphism

$$X(R) \times_{X(\mathrm{Cone}[\mathrm{Nil}(R)\otimes_R L \to R])} U(\mathrm{Cone}[\mathrm{Nil}(R) \otimes_R L \to R]) \simeq U(R).$$

This is equivalent to unique infinitesimal lifting for the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{Cone}[\mathrm{Nil}(R) \otimes_R L \to R]) & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(R) & \longrightarrow & X. \end{array}$$

For (c), note that since $U^{\bullet,dR,+}$ are open subfunctors by (b), we have an inclusion

$$\mathrm{colim}\, U^{\bullet,dR,+} \hookrightarrow X^{dR,+}.$$

We hope to check this is surjective. There is a diagram (2.1) gives

$$\begin{array}{ccc} \mathrm{colim}\, U^\bullet \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & \mathrm{colim}\, U^{\bullet,dR,+} \\ \simeq\downarrow & & \downarrow \\ X \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & X^{dR,+} \end{array}$$

By Remark 2.9, we know the horizontal maps are surjective, so the right vecrtical map must also be surjective, as desired. □

Now, we can finally:

Proof of Theorem 2.10. We proceed in three steps:

(1) We claim $\mathcal{H}_{dR,+}(X) \in \mathcal{D}(\mathbb{A}^1/\mathbb{G}_m, \mathcal{O})$ is quasi-coherent and t-complete, and moreover its restriction to $B\mathbb{G}_m$ agrees with the natural pushforward map along $X^{Hodge} \to B\mathbb{G}_m$. By Lemma 2.11(c) it suffices to check when there is an étale map $f\colon X \to \mathbb{A}^n$. By (b) there is a cartesian square

$$\begin{array}{ccc} X \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & X^{dR,+} \\ \downarrow & & \downarrow \\ \mathbb{A}^n \times \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & (\mathbb{A}^n)^{dR,+}. \end{array}$$

The bottom map is a $G = \mathbf{V}(\widehat{\mathcal{O}(-1)})^n$-torsor. Indeed, by (a) it suffices to consider $n = 1$, in which case it follows by definition. Now the top horizontal map must be a G-torsor as well, so we have an isomorphism

$$X^{dR,+} \simeq (X \times \mathbb{A}^1/\mathbb{G}_m)/G.$$

Now by Remark 1.21 the pushforward $\mathcal{H}_{dR,+}(X)$ is quasi-coherent and is compatible with base change. Moreover, t-completeness follows from transporting completeness along the equivalence $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathcal{D}_{fil}(k)$, as in Theorem 1.9(4).

(2) By derived deformation theory, there is an isomorphism

$$X^{Hodge} \simeq B\mathbf{V}(\widehat{T_{X/k}(-1)}),$$

where $T_{X/k}(-1) = pr_1^* T_{X/k} \otimes pr_2^* \mathcal{O}_{B\mathbb{G}_m}(-1)$. Indeed, given:

- a finite type k-scheme X;
- an animated k-algebra R;
- a map $\eta\colon \operatorname{Spec}(R) \to X$ of derived k-schemes; and
- a square-zero extension $R' \to R$ in animated k-algebras by $N \in \mathcal{D}^{\leq 0}(R)$,

the fiber of the map $X(R') \to X(R)$ over $\eta \in X(R)$ is a torsor for

$$\operatorname{Der}(\mathcal{O}_X, \eta_* N) \simeq \operatorname{Map}_R(\eta^* L_{X/k}, N).$$

When X is smooth and $N = L[1] \in \mathcal{D}^{\leq -1}(R)$, we have

$$\operatorname{Map}_R(\eta^* L_{X/k}, N) \simeq B(\eta^* T_{X/k} \otimes_R L).$$

When furthermore $R' \to R$ is split, the fiber of $X(R') \to X(R)$ is a split torsor, canonically identified with $B(\eta^* T_{X/k} \otimes_R L)$.

Let us apply this to the square-zero extension

$$R \oplus \operatorname{Nil}(R)(-1)[1] \to R$$

by $\operatorname{Nil}(R)(-1)[1]$, so $X^{Hodge} \to X$ is a split torsor with fibers

$$B(\eta^* T_{X/k} \otimes_R \operatorname{Nil}(R) \otimes_R L),$$

i.e.,

$$X^{Hodge} \simeq B\mathbf{V}(\widehat{T_{X/k}(-1)}).$$

Now, Proposition 1.19, gives an equivalence

$$\mathcal{D}_{qc}(X^{Hodge}) \simeq \mathcal{D}_{qc}(\mathbf{V}(\Omega_{X/k}(1))),$$

sending $\mathcal{O}_{X^{Hodge}}$ to $\mathcal{O}_{X\times\mathbb{G}_m}$. To compute the pushforward $\pi_{X,*}\mathcal{O}_{X^{Hodge}}$, note that under the equivalence this is equivalent to computing

$$RHom_{\mathbf{V}(\Omega_{X/k}(1))}(\mathcal{O}_{X\times\mathbb{G}_m}, \mathcal{O}_{X\times\mathbb{G}_m}).$$

But the Koszul resolution provides a quasi-isomorphsim between $\mathcal{O}_{X\times\mathbb{G}_m}$ and

$$[\cdots \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))} \otimes_{\mathcal{O}_{X\times B\mathbb{G}_m}} \wedge^2 T_{X/k}(-2) \to \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))} \otimes_{\mathcal{O}_{X\times B\mathbb{G}_m}} T_{X/k}(-1) \to \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))}],$$

which provides an isomorphism

$$\begin{aligned} RHom_{\mathbf{V}(\Omega_{X/k}(1))}(\mathcal{O}_{X\times\mathbb{G}_m}, \mathcal{O}_{X\times\mathbb{G}_m}) &\simeq RHom_{\mathbf{V}(\Omega_{X/k}(1))}([\cdots \to \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))} \otimes T_{X/k}(-1) \to \mathcal{O}_{\mathbf{V}(\Omega_{X/k}(1))}], \mathcal{O}_{X\times\mathbb{G}_m}) \\ &\simeq RHom_{X\times\mathbb{G}_m}(\cdots \xrightarrow{0} T_{X/k}(-1) \xrightarrow{0} \mathcal{O}_{X\times\mathbb{G}_m}, \mathcal{O}_{X\times\mathbb{G}_m}) \\ &\simeq \bigoplus_i R\Gamma(X, \Omega^i_{X/k}[-i])(i). \end{aligned}$$

(3) We have a presheaf $\mathcal{F}\colon U \mapsto \mathcal{H}_{dR,+}(U)$ on X valued in $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathcal{D}_{fil}(k)$, which is a sheaf by (c). Moreover, by the first two parts, the sheaf $\mathcal{F}$ lies in the heart of the Beilinson t-structure. Thus, it is represented by a chain complex, which by (2) must be of the form

$$\mathcal{O}_X \xrightarrow{\delta} \Omega^1_{X/k} \xrightarrow{\delta} \Omega^2_{X/k} \xrightarrow{\delta} \cdots,$$

for some differentials δ, equipped with the stupid filtration. To prove the theorem, we need only check that δ are the de Rham differentials. By Lemma 2.11, it suffices to show the case $X = \mathbb{A}^1$. In this case, $X^{dR,+} = \mathbb{G}_a^{dR,+}$ and by Remark 1.21 the cohomology $\mathcal{H}_{dR,+}(X)$ is computed by the graded $k[t]$-complex

$$k[t,x] \xrightarrow{t\frac{d}{dx}} k[t,x](1),$$

since the differential is $\frac{d}{dx}$ on the non-filtered objects. Thus translating to filtered objects, we see that $\delta = \frac{d}{dx}$. □

The stack $X^{dR,+}$ not only geometrizes de Rham cohomology, but it also geometrizes the category of vector bundles with flat connections.

Remark 2.12. By pullback along the map $X \times \mathbb{A}^1/\mathbb{G}_m \to X^{dR,+}$ from Remark 2.9, the category $\mathrm{Vect}(X^{dR,+})$ of vector bundles on $X^{dR,+}$ is identified with the category of triples (E, ∇, F^*) where:

- E is a vector bundle on X;
- $\nabla\colon E \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} E$ is a flat connection; and
- F^* if a finite filtration of E by submodules satisfying Griffits transversality:

$$\nabla(F^i) \subset \Omega^1_{X/k} \otimes_{\mathcal{O}_X} F^{i-1}.$$

Similarly, the pullback along $X \times B\mathbb{G}_m \to X^{Hodge}$ identifies $\mathrm{Vect}(X^{Hodge})$ with the category of graded Higgs bundles, i.e., graded vector bundles $M = \bigoplus_i M_i$ together with a Higgs field $\Theta\colon M \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} M$ (i.e., such that $\Theta \wedge \Theta = 0$) taking M_i to $\Omega^1_{X/k} \otimes_{\mathcal{O}_X} M_{i+1}$. Under this description, $\mathrm{Vect}(X^{dR,+}) \to \mathrm{Vect}(X^{Hodge})$ simply takes associated graded.

3. Linear algebra outside of characteristic 0

The key tool in the stacky description of de Rham cohomology in characteristic zero was Remark 1.20 and the relative analog, Remark 1.21. To extend to arbitrary characterstic, we need:

Definition 3.1. Let $\mathbb{G}_a^\#$ be the PD[1]-hull of the origin in $\mathbb{G}_a$ over $\mathbb{Z}$. Explicitly, letting $\mathbb{G}_a = \operatorname{Spec}\mathbb{Z}[t]$, we let

$$\mathbb{G}_a^\# := \operatorname{Spec}\left(\mathbb{Z}\big[t, \frac{t^2}{2!}, \frac{t^3}{3!}, \cdots\big]\right).$$

Thus, an R-point of $\mathbb{G}_a^\#$ is exactly an element $x \in R$ with a compatible system of divided powers, i.e., elements $\{x_n\}_{n\geq 1}$ of R such that $x_1 = x$ and

$$x_m x_n = \binom{m+n}{m} x_{m+n}.$$

There is a natural map $\mathbb{G}_a^\# \to \mathbb{G}_a$ which is an isomorphism upon base change to $\mathbb{Q}$, and the group law on $\mathbb{G}_a$ induces a group law on $\mathbb{G}_a^\#$, since

$$\frac{(x+y)^n}{n!} = \sum_{i+j=n} \frac{x^i}{i!}\frac{y^j}{j!}.$$

Moreover $\mathbb{G}_a^\#$ is a $\mathbb{G}_a$-module since

$$\frac{(xy)^n}{n!} = x^n \frac{y^n}{n!},$$

so $\mathbb{G}_a^\#$ is a quasi-ideal in $\mathbb{G}_a$.

Example 3.2. If R is $\mathbb{Z}$-flat (or equivalently, torsion-free), then $\mathbb{G}_a^\#(R) \to \mathbb{G}_a(R) = R$ is injective, with image consisting of $x \in R$ such that $x^n \in n! \cdot R$. In particular, when $R = \mathbb{Z}_p$, we have $\mathbb{G}_a^\#(\mathbb{Z}_p) = p\mathbb{Z}_p \subset \mathbb{Z}_p$, since $x^n \in n!\mathbb{Z}_p$ is equivalent to $v_p(x^n) \geq v_p(n!)$.

Example 3.3. When R is a $\mathbb{F}_p$-algebra, $\mathbb{G}_a^\#(R) \to \mathbb{G}_a(R) = R$ is injective if and only if R is reduced. Indeed, the kernel consists of a system of divided powers $\{x_n\}_{n\geq 1}$ such that $x_1 = 0$. If R is reduced, then $x_n = 0$, since

$$x_n^p = \frac{(pn)!}{n!^p} x_{np} = 0.$$

On the other hand, suppose $\mathbb{G}_a^\#(R) \to R$ and $x \in R$ is such that $x^2 = 0$. Then

$$x_n = \begin{cases} x & n = p \\ 0 & n \neq p \end{cases}$$

is a compatible system of divided powers with $x_1 = 0$.

Definition 3.1 globalizes:

Definition 3.4. If E is a vector bundle on a scheme X, we let $\mathbf{V}(E)^\#$ denote the PD-hull of the 0-section in $\mathbf{V}(E)$. Then $\mathbb{V}(E)^\#$ is a $\mathbb{G}_a$-module scheme over X and the map $\mathbf{V}(E)^\# \to \mathbf{V}(E)$ is a $\mathbb{G}_a$-module homomorphism.

Now, we have the following generalization of (1.1), with the same proof:

Proposition 3.5. *There is a natural monoidal equivalence*

$$\mathcal{D}_{qc}(B\mathbb{G}_a^\#) \simeq \mathcal{D}_{qc}(\widehat{\mathbb{G}}_a),$$

where the right-hand side has the convolution product. Moreover:

[1]PD stands for the French *puissances divisées* for divided powers.

(1) There is a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{qc}(B\mathbb{G}_a^{\#}) & \xrightarrow[\simeq]{\Phi} & \mathcal{D}_{qc}(\widehat{\mathbb{G}}_a) \\ & {\scriptstyle \pi^*}\searrow \quad \swarrow{\scriptstyle R\Gamma} & \\ & \mathcal{D}(R), & \end{array}$$

where $\pi\colon \operatorname{Spec}(\mathbb{Z}) \to B\mathbb{G}_a^{\#}$ *is the standard map and* $R\Gamma$ *denotes the local cohomology at* 0.

(2) The equivalence is t-exact, with respect to the standard t-structure on the LHS and the torsion t-structure on the RHS.

Remark 3.6. The torsion t-structure is given by the following general construction: given a commutative ring R with finitely generated ideal I, let $\mathcal{D}_{I-comp}(R)$ (resp., $\mathcal{D}_{I-tors}(R)$) be the full subcategories of $\mathcal{D}(R)$ spanned by derived I-complete (resp., I^∞-torsion) R-complexes. Then the functor taking local cohomology $R\Gamma_I(-)$ and $(-)^\wedge_I$ give an equivalence $\mathcal{D}_{I-comp}(R) \simeq \mathcal{D}_{I-tors}(R)$. The standard t-structure on the torsion side induces a t-structure on the complete side, called the "torsion t-structure." In our situation $R = \mathbb{Z}[\![t]\!]$ and $I = (t)$. Then the equivalence is

$$\mathcal{D}_{t-comp}(\mathbb{Z}[\![t]\!]) \simeq \mathcal{D}_{t-tors}(\mathbb{Z}[\![t]\!])$$

$$M \mapsto R\Gamma_t(M) = \operatorname{Fib}\big(M \to M \otimes^{\mathbb{L}}_{\mathbb{Z}[\![t]\!]} \mathbb{Z}(\!(t)\!)\big) = M \otimes^{\mathbb{L}} \mathbb{Z}(\!(t)\!)/\mathbb{Z}[\![t]\!][-1]$$

$$\widehat{M} = \lim_n M \otimes^{\mathbb{L}}_{\mathbb{Z}[\![t]\!]} \mathbb{Z}[t]/t^n \leftarrow M.$$

For example, $\mathbb{Z}[t]/t^N \mapsto \mathbb{Z}[t]/t^N$. In general the functor is non-trivial:

$$\begin{aligned} \widehat{\mathbb{Z}(\!(t)\!)/\mathbb{Z}[\![t]\!]} &\simeq \lim_n \mathbb{Z}(\!(t)\!)/\mathbb{Z}[\![t]\!] \otimes^{\mathbb{L}} \mathbb{Z}[t]/t^n \\ &\simeq \lim_n \operatorname{Cone}\big(\mathbb{Z}(\!(t)\!)/t^{-n}\mathbb{Z}[\![t]\!] \to \mathbb{Z}(\!(t)\!)/\mathbb{Z}[\![t]\!]\big) \\ &\simeq \lim_n t^{-n}\mathbb{Z}[\![t]\!]/\mathbb{Z}[\![t]\!][1] \\ &\simeq \mathbb{Z}[\![t]\!][1]. \end{aligned}$$

There is also a relative analog:

Proposition 3.7. *Let X be a scheme and let E be a vector bundle on X. Then there is a t-exact monoidal equivalence*

$$\mathcal{D}_{qc}(B\mathbf{V}(E)^{\#}) \simeq \mathcal{D}_{qc}(\widehat{\mathbb{V}(E^\vee)})$$

compatible with forgetful functors.

References

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