Introduction to prismatic cohomology

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In this introductory talk, I'll sketch Bhatt–Scholze's definition of prismatic cohomology, and talk about several interesting applications of the theory. This is a quickly growing field and if you have a better idea of what's going on, feel free to interrupt me at any moment.

Before I get started, here are some things I personally want to get out of this seminar. Feel free to make suggestions about where you'd like this seminar to go.

1. I want to have an (at least somewhat rigorous) understanding of different approaches of defining prismatic cohomology, and why they're equivalent;
2. I want to be able to translate between the topological viewpoint and the algebraic viewpoint e.g. in Bhatt's lectures;
3. Motivate myself to do more calculations in homotopy theory.

1 Background

1.1 Motivation. In complex geometry, we study smooth projective varieties $X/\mathbb{C}$. Such an object admits many different points of view:

- As a topological space it has singular (co)homology groups $H^*(X;\mathbb{Z})$.
- As a smooth real manifold it has de Rham cohomology $H^*_{\mathrm{dR}}(X;\mathbb{C})$.
- As a Kähler manifold it has Hodge decomposition $H^{p,q}(X)$.

Classical theorems give us canonical isomorphisms between these. In other words, we can see lots of topological information of X through geometry, and vice versa.

Unfortunately, integrating forms in char. 0 does not see the torsion in singular homology. If we want to both have torsion information and geometric tools, we would like to be in mixed characteristic. For example, let X now be a smooth projective variety over $\mathbb{Z}_p$. By base change we get a special fiber $X_{\mathbb{F}_p}$ and a generic fiber $X_{\mathbb{C}_p}$. Note that $\mathbb{C}_p \simeq \mathbb{C}$ abstractly as fields so the latter is really a geometric object. A principle developed in the 20th century is that the p-torsion topological information in $X_{\mathbb{C}_p}$ can be seen by the algebraic differential forms on $X_{\mathbb{F}_p}$. Sasha told me about something Fontaine knew already before prismatic cohomology, the statement being that in degree less than p, the étale and de Rham cohomologies (both with $\mathbb{Z}_p$ coefficients) are noncanonically isomorphic.

1.2 An example application. Primsatic cohomology associates to X a finitely generated $\mathbb{F}_p[[t]]$-module $H^*_{\Delta}(X;\mathbb{F}_p)$. It can be seen as a deformation of the de Rham cohomology. It satisfies the comparison isomorphisms (in the derived sense)

- $H^*_{\Delta}(X;\mathbb{F}_p)/t \simeq H^*_{\mathrm{dR}}(X_{\mathbb{F}_p})$;

- $H^*_{\Delta}(X;\mathbb{F}_p)[t^{-1}] \simeq H^*(X_{\mathbb{C}_p};\mathbb{F}_p)((t))$.

Consequently, by the structure theorems of finitely generated modules over a PID, we see the *torsion inequality*

$$\dim_{\mathbb{F}_p} H^n(X_{\mathbb{C}_p};\mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H^n_{\mathrm{dR}}(X_{\mathbb{F}_p}).$$

This inequality can be strict, and it cannot be shown by directly putting one as a subquotient of the other.

This object has several different distinct incarnations. The original construction in BMS1 uses perfectoid spaces and the almost purity theorem. In BMS2 there was a construction using the motivic filtration on topological periodic homology (but this only gives the Nygaard-completed version). I'll focus on the construction using prisms, which is the most elementary. There is also a stacky approach by Drinfeld and Bhatt–Lurie.

This is the end of the motivational section so people should comment now.

1.3 δ-rings. A δ-ring is just a ring A together with a map $\delta : A \to A$ satisfying a bunch of axioms which make $\phi(x) = x^p + p\delta(x)$ a ring homomorphism. Examples: $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Z}_p[x]$, $\mathbb{Z}_p[[x]]$, ...

Maybe a good way to think about this is through Witt vectors. For each ring A there is a ring $W(A)$ the p-typical Witt vectors, whose elements can be expressed as infinite sequences $(x_0, x_1, \dots)$ with addition and multiplication such that the *ghost map* $W(A) \to \prod_{\mathbb{N}} A$, $(x_0, x_1, \dots) \mapsto (\sum_{i=0}^n p^i x_i^{p^{n-i}})_n$ is a natural transformation of functors of rings. When A is a perfect $\mathbb{F}_p$-algebra, $W(A)$ should be seen as the universal deformation of A over $\mathbb{Z}_p$, analogous to the situation of $k[[x]]$ for k. Let $W_n(A)$ be the subring of sequences where $x_i = 0$ for $i \geq n$. This is analogous to $k[\varepsilon]/\varepsilon^n$. We see that in $W_2(A)$ the ghost map is $(x_0, x_1) \mapsto (x_0, x_0^p + px_1)$. Thus, a δ-ring structure on A is a section $A \to W_2(A)$ of the zeroth ghost map $W_2(A) \to A$. This is like the definition of a derivation on A being a section to the projection $A[\varepsilon]/\varepsilon^2 \to A$.

Another view, when A is p-torsion free, is that we have the fiber product of rings $W_2(A) = A \times_{A/p} A$, where one of the maps is just mod p and the other is mod p followed by Frobenius. So $\operatorname{Spec} W_2(A)$ is two copies of $\operatorname{Spec} A$ glued along Frobenius on $\operatorname{Spec} A/p$. A δ-ring structure is then a retract $\operatorname{Spec} W_2(A) \to \operatorname{Spec} A$. For general A this has to be interpreted in a derived way.

1.4 Derived completeness. This is just a technical point that maybe we'll gloss over now. Let A be a ring and $I \subset A$ a finitely generated ideal. Classically, we define the I-completion of an A-module M to be $M^\wedge_I = \lim_n M/I^n M$. However, this notion is not so well-behaved. For example, classically I-complete modules do not form an abelian subcategory, and completions of flat A-modules are not necessarily flat. The replacement notion is:

1.5 Definition. An A-module M is *derived I-complete* if for any $f \in I$, $R\operatorname{Hom}_A(A_f, M) \simeq 0$ (equivalently Hom and Ext^1 vanish).

1.6 Proposition. *Good properties of derived completeness:*

1. *M is classically I-complete iff M is derived I-complete and I-separated.*
2. *(Derived Nakayama's lemma) For M derived I-complete, $M = IM \implies M = 0$.*
3. *The category of derived I-complete modules is an abelian subcategory (but not AB5).*
4. *The inclusion from derived I-complete modules to modules admits a left adjoint, called the derived I-completion $(-)^\wedge_I$.*
5. *When $I = (f)$ is principal this is concretely $M^\wedge_f = \operatorname{Ext}^1_A(A_f/A, M)$. Moreover, if M has bounded f-torsion then this coincides with classical f-completion.*
6. *If A is derived I-complete, then it is I-local, and every finitely presented A-module is derived I-complete.*

2 Prismatic cohomology

2.1 Prisms. A *prism* is a pair (A, I) where A is a δ-ring, I an invertible ideal (i.e. locally principal generated by non-zero-divisor), $p \in (I, \phi(I))$, and A is derived (p, I)-complete. In particular A is (p, I)-local.

In many cases we can reduce to the local situation: $I = (f)$ is principal ("oriented"), we have $p \in (f, \phi(f)) \iff \delta(f)$ is a unit, which is equivalent to f being a *distinguished element*. This is roughly saying that f vanishes to first order in A/p according to the derivation δ.

Say A is *bounded* if A/I has bounded p-torsion (which implies that A is classically (p, I)-complete), and *perfect* if ϕ is an automorphism. A map of prisms $(A, I) \to (B, J)$ is a map of δ-rings $f : A \to B$ such that $f(I) \subset J$. Exercise: this necessarily implies $IB = J$.

2.2 Example. (1) (Crystalline prisms) A is a p-complete, p-torsion-free δ-ring, and $I = (p)$. For example, $A = W(R)$ where R is a perfect $\mathbb{F}_p$-algebra with characteristic p, then the Frobenius on A which shifts the ghost coordinates $(w_0, w_1, \dots) \mapsto (w_1, w_2, \dots)$ lifts the Frobenius on $R = A/p$. It is a general property that in this case $A = W(R)$ satisfies the assumptions, and furthermore it is bounded.

(2) $A = \mathbb{Z}_p[[t]]$ with $\phi(t) = t^p$, and $I = (t - p)$. Then $A/I = \mathbb{Z}_p$.

(3) $A = \mathbb{Z}_p[[q - 1]]$ with $\phi(q) = q^p$, and $I = [p]_q = 1 + q + \cdots + q^{p-1}$. Then $A/I = \mathbb{Z}_p[\zeta_p]$.

(4) Perfect prisms (A, I) are equivalent to perfectoid rings A/I. I don't know much about what those are so someone should tell me.

2.3 Prismatic envelope. For any prism (A, I), the forgetful functor

$$\{\text{prisms } (B, J) \text{ over } (A, I)\} \to \{\delta\text{-pairs } (B, J) \text{ over } (A, I)\}$$

has a left adjoint called the prismatic envelope.

2.4 The relative prismatic site. Let (A, I) a bounded prism. We will just do the affine situation. Let R be a (formally smooth) ring over A/I. An object in the relative prismatic site $(R/A)_{\Delta}$ (I guess opposite) is a bounded prism (B, J) over (A, I) together with a map of A/I-algebras $R \to B/J$. We'll give it the "indiscrete topology", meaning that the only covers are isomorphisms. So every presheaf is a sheaf. (In the non-affine case, we'll have to use the flat topology.) The struture sheaf $\mathcal{O}_{\Delta}$ is $(B, J) \mapsto B$, and the reduced structure sheaf $\overline{\mathcal{O}_{\Delta}}$ is $(B, J) \mapsto B/J$.

In general, there is no terminal object in this category. Maybe a good example is $R = (A/I)[x]^{\wedge}_p$. There exists a bounded prism (B, J) where $B = A[x]^{\wedge}_{(p,I)}$ (classical completion) such that $R \simeq B/J$, but it is not terminal. But there is always a weakly terminal object: let B' be the free δ-ring over A on the set of elements of R, and let $J' = \ker(B' \to R)$, then let (B, J) be the prismatic envelope of (B', J'). This is weakly terminal.

2.5 Computing cohomology in the indiscrete topology. Suppose X is a weakly terminal object, then $R^*\Gamma(\mathcal{C}; F)$ can be computed by the Čech complex of X:

$$0 \to F(X) \rightrightarrows F(X \times X) \substack{\to\\\to\\\to} \cdots$$

2.6 Relative prismatic cohomology. Define $\Delta_{R/A} = R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$ and the Hodge–Tate complex $\overline{\Delta_{R/A}} = R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}_{\Delta}})$. They canonically live in $\mathcal{D}(A)$ but admit no obvious models.

$\Delta_{R/A}$ is a derived (p, I)-complete commutative algebra object. In fact, it admits the structure of an $\mathbb{E}_\infty$-algebra in the ∞-category, but it cannot be modeled by a CDGA. $\Delta_{R/A}$ also naturally carries a Frobenius $\phi : \phi_A^* \Delta_{R/A} \to \Delta_{R/A}$.

By the previous section we can get a hold on $\Delta_{R/A}$ using the above cosimplicial ring. This should also present some stack?

Draw the picture for the following theorem.

2.7 Theorem. *Let A as in example 2. Suppose R is the p-completion of a smooth $\overline{A}$-algebra.*

1. *(Hodge–Tate comparison) There is a canonical quasi-isomorphism $\widehat{\Omega}^*_{R/\overline{A}} \to H^*(\overline{\mathbb{\Delta}}_{R/A})\{*\}$ where the latter has the Bockstein differential. This is specialization along $t = p$.*

2. *(de Rham comparison) There is a canonical quasi-isomorphism $\widehat{\Omega}^*_{R_{\mathbb{F}_p}/\mathbb{F}_p} \to (\phi^* \mathbb{\Delta}_{R/A}) \otimes_A \mathbb{F}_p$. This is specialization at $t = p = 0$. (In fact this lifts to specializing at $t = p^p$.)*

3. *(étale comparison) There is a canonical quasi-isomorphism $H^*_{\text{ét}}(\operatorname{Spec} R_{\mathbb{C}_p}; \mathbb{F}_p) \simeq (\mathbb{\Delta}_{R/A} \otimes \overline{\mathbb{F}_p((t))})^{\phi-1}$ (the derived fixed points).*

4. *The map ϕ becomes a quasi-isomorphism after inverting $t - p$.*

2.8 The absolute prismatic site and the Nygaard filtration. If we do not fix a base prism A/I and instead just take the site of bounded prisms (B, J) admitting a map $\operatorname{Spf} R \to B/J$, we get the absolute site, and we can similarly define absolute prismatic cohomology. The right way to do this is probably via prismatization, which will be a later talk.

The absolute $\mathbb{\Delta}_R$ carries a canonical filtration. Roughly this plays the same role as the Hodge filtration for de Rham cohomology. The easiest way to define this is probably through topology and arises from the homotopy fixed point spectral sequence on TC^-. This will also be mentioned in a later talk.

2.9 q-de Rham complexes. As a final application, consider a smooth $\mathbb{Z}_p$-algebra R, and a choice of coordinates (étale map $\mathbb{A}^n_{\mathbb{Z}_p} \to R$), we can define a q-deformed version of the de Rham complex of $R/\mathbb{Z}_p$. The complex depends wildly on the choice of coordinates, but Scholze conjectured that it is a canonical object in the derived category. A surprising application of prismatic cohomology (with base prism example 3) proved this conjecture: in fact after completing R there is a canonical quasi-isomorphism $q\Omega^*_{R/\mathbb{Z}_p} \simeq \mathbb{\Delta}_{R \otimes_{\mathbb{Z}_p} \overline{A}/A}$, and the latter does not depend on the coordinates. Maybe do an example calculation.