Def Prism is a δ-ring A with Cartier divisor $I \subset A$ (Zariski locally on A $I=(d)$ for a non zero-div. $d \in A$)

s.t.

1) A is (p, I)-complete

2) $p \in I + \varphi_A(I)$

(if $I=(d)$, equivalently, $\delta(d)$ is a unit)

$\varphi_A(x) = x^p + p\,\delta(x)$

δ-structure $A \xrightarrow{\delta} A$ $\leftrightarrow$ ring map $A \xrightarrow{\delta} W_2(A)$, $a \mapsto (a, \delta(a))$, splitting $W_2(A) \to A$ $\rightsquigarrow$ δ-ring map $A \longrightarrow W(A)$, $a \mapsto (a, \delta(a), \delta_2(a), \ldots)$

$X / A/I$ p-adic formal scheme

prismatic site $(X/A)_{\Delta} = \left\{ \begin{matrix} B \longrightarrow B/IB \\ \uparrow \quad\quad \uparrow \\ A \longrightarrow A/I \end{matrix} \text{ with } \operatorname{Spf} B/IB \to X \text{ over } \operatorname{Spf} A/I \right\}$, with B a δ-ring map, s.t. $B \supset IB$ is a prism with bounded p-torsion

$A \longrightarrow B$ is a cover if it is (p, I)-complete flat.

$\mathcal{O}_{\Delta}(B) = B$

For affine $X = \operatorname{Spf} R$

$$R\Gamma_{\Delta}(R/A) = \lim_{B \in (R/A)_{\Delta}} B$$

Thm 1 X formally smooth $/A/I$

$$\varphi_A^* R\Gamma_{\Delta}(X/A) \otimes_A^{\mathbb{L}} A/I \simeq R\Gamma_{dR}(X/A/I)$$

Ex $A = \mathbb{Z}_p[[u]]$ $A/I = \mathcal{O}_K$ $K \supset \mathbb{Q}_p$ finite ext'n

$\forall X/\mathcal{O}_K$ smooth $R\Gamma_{dR}(X/\mathcal{O}_K)$ descends along $\mathbb{Z}_p[[u]] \xrightarrow{u \mapsto u^p} \mathbb{Z}_p[[u]] \longrightarrow \mathcal{O}_K$

for $k = \mathbb{Q}_p(p^{1/p})$ this implies that $R\Gamma_{dR}(X/\mathcal{O}_k)$ descends to $\mathbb{Z}_p$.

Pf Assume $I = (p) \subset A$ $R/_{A/p}$ smooth

Will prove $\varphi_A^* R\Gamma_{\Delta}(R/A) \simeq R\Gamma_{cris}(R/A) \simeq R\Gamma_{dR}(\tilde{R}/A)$

$$\begin{array}{ccc} \circlearrowleft & & \circlearrowleft \\ \varphi_{\Delta} & = & \varphi_R^* \end{array}$$

if $\tilde{R}/A$ is a flat lift

This implies de Rham comparison in general

$$\begin{array}{ccc} B & \longrightarrow & B/p \nwarrow \\ \uparrow \circlearrowright \varphi_B & & \quad R \\ & & \nearrow \\ A & \longrightarrow & A/p \end{array}$$

$B/\mathbb{Z}_p$ flat, p-complete

locally free prismatic crystals on $(R/A)_{\Delta}$ $:= \left\{ \begin{array}{l} \text{projective } B\text{-module } \mathcal{E}_B \\ \text{for every } B + \\ \mathcal{E}_B \otimes_B B' \simeq \mathcal{E}_{B'} \\ \text{compatible with composition} \end{array} \right\}$

Prop $R/\mathbb{F}_p$, $A = \mathbb{Z}_p$ prismatic crystals on $R/\mathbb{Z}_p$ $=$ Crystals on R $=$ projective $\hat{R}$-modules M with $\nabla : M \to M \otimes \Omega^1_{\hat{R}/\mathbb{Z}_p}$ s.t. ∇ mod p has loc. nilp. p-curvature

Pf Given $\mathcal{E}$, how to recover M/p?

$$\begin{array}{ccc} W(R) & \longrightarrow & W(R)/p \\ \uparrow & & s \uparrow \quad [r^p] \\ & & \quad\ \uparrow \\ \mathbb{Z}_p & \longrightarrow \mathbb{F}_p \nearrow & R \quad\ r \end{array}$$

$W(R) \xrightarrow{F} W(R)/p$

$[a] \mapsto [a^p]$

factors through R because $FV = p$.

$M/p := \mathcal{E}_{W(R)} \otimes_{W(R)} R$

• If $\tilde{R}/\mathbb{Z}_p$ is a lift with Frobenius $\varphi_{\tilde R} \circlearrowright \tilde R$ then

$$\begin{array}{ccc} \tilde R & \longrightarrow & R \\ \uparrow & & \| \; R \\ \mathbb{Z}_p & \longrightarrow & \mathbb{F}_p \nearrow \end{array}$$

$M := \varphi_{\tilde R}^* \mathcal{E}_{\tilde R}$ consistent with above formula for M/p

$$\tilde R \xrightarrow{\delta_{\tilde R}} W(\tilde R) \to W(R) \to W(R)/p$$
$$\tilde R \longrightarrow R \xrightarrow{\delta_{\tilde R} \bmod p} W(R)/p, \qquad R \,\|\, R$$
$$R \xrightarrow{\delta} W(R)/p, \quad R \xrightarrow{\varphi_R} R \xrightarrow{\delta_{\tilde R} \bmod p} W(R)/p$$

<u>Recall</u>: $\left\{ \begin{array}{l} \text{modules } M/\hat R \\ \text{with flat connection} \end{array} \right\}_{\text{nilp } p\text{-curv.}} \simeq \left\{ \begin{array}{l} \text{modules } N/\hat R \\ \text{with } p\text{-connection} \end{array} \right\}_{\nabla \bmod p \text{ nilpotent}}$

$\nabla : N \to N \otimes \Omega^1_{\hat R}$

$$\left(\varphi_{\hat R}^* N, \frac{\varphi_{\hat R}^* \nabla}{p}\right) \longleftarrow (N, \nabla) \qquad \nabla(fn) = f\nabla(n) + p\, df \otimes n$$

Assume $R = \mathbb{F}_p[x]$ $\quad \hat R = \mathbb{Z}_p\langle x \rangle \quad \varphi(x) = x^p$

need to produce a connection on

$$\mathbb{Z}_p\langle x\rangle \longrightarrow \mathbb{F}_p[x] \qquad \varphi_{\hat R}^* \mathcal{E}_{\mathbb{Z}_p\langle x\rangle} =: M$$

<u>Recall</u> crystal structure on $V/\mathbb{Z}_p\langle x\rangle$ $= p_1^* V \simeq p_2^* V$ on

$$\begin{array}{ccc} \mathbb{Z}_p\langle x_1, x_2\rangle\left\{\frac{x_1 - x_2}{p}\right\} \circlearrowleft \varphi & \longrightarrow & \mathbb{F}_p\left[x, \frac{x_1 - x_2}{p}, \dots\right] \\ \uparrow & & \uparrow \\ \mathbb{Z}_p\langle x_1, x_2\rangle & \longrightarrow & \mathbb{F}_p[x_1, x_2] \\ q_1 \uparrow \uparrow q_2 & & \uparrow\uparrow \\ \mathbb{Z}_p\langle x\rangle & \longrightarrow & \mathbb{F}_p[x] \end{array}$$

$$\frac{x_1 - x_2}{p} \longmapsto \frac{x_1^p - x_2^p}{p} = \frac{x_1 - x_2}{p}\cdot(\dots)$$

$$\mathbb{Z}_p\left\langle x_1, x_2, \frac{(x_1 - x_2)^n}{n!} \;\middle|\; n \in \mathbb{N}\right\rangle \quad p_1 \nearrow \nearrow p_2 \quad \mathbb{Z}_p\langle x\rangle \qquad p_i(x) = x_i$$

satisfying cocycle condition.

for $x \in A$ $\quad A\left\{\frac{x}{p}\right\} :=$ initial δ-ring $A \to B$ s.t. $x \in pB$.

prismatic crystal structure $\rightsquigarrow$ $q_1^* \mathcal{E}_{\mathbb{Z}_p\langle x\rangle} \simeq \mathcal{E}_{\mathbb{Z}_p\langle x_1, x_2\rangle\left\{\frac{x_1 - x_2}{p}\right\}} \simeq q_2^* \mathcal{E}_{\mathbb{Z}_p\langle x\rangle}$. $\quad (*)$

Base changing along $\mathbb{Z}_p\langle x_1, x_2\rangle\left\{\frac{x_1 - x_2}{p}\right\} \xrightarrow{x_1 \mapsto x_1^p,\; x_2 \mapsto x_2^p} \mathbb{Z}_p\langle x_1, x_2\rangle\left\{\frac{x_1^p - x_2^p}{p}\right\}$

Key lemma $A/\mathbb{Z}_p$ flat $\varphi_A \circlearrowleft A$ ~~If for~~ For some $x \in A$ suppose $x^p \in pA$ then $\forall n$ $x^n \in n!A$

Pf Equivalently, need $x^{p^k} \in p^{p^{k-1}+\dots+p+1}A$. Will do $k=2$.

$x^p \in pA \iff \varphi_A(x) \in pA$.

$$\left(\frac{x^p}{p}\right)^p = \frac{x^{p^2}}{p^p} \equiv_{\text{mod } p} \varphi_A\left(\frac{x^p}{p}\right) = \frac{\varphi_A(x)^p}{p} = \frac{(x^p + p\delta(x))^p}{p}$$ divisible by p^{p-1}

$\Rightarrow x^{p^2} \in p^{p+1}A$. ∎

Therefore

$\mathbb{Z}_p\langle x_1, x_2\rangle\left\{\frac{x_1^p - x_2^p}{p}\right\} \subset \mathbb{Z}_p\langle x_1, x_2, \frac{(x_1-x_2)^n}{n!}\rangle$ is equality.

$\qquad \parallel$

$\mathbb{Z}_p\langle x_1, x_2\rangle\left\{\frac{(x_1-x_2)^p}{p}\right\}$

and base change of $(*)$ gives $p_1^* M \simeq p_2^* M$. ∎

Prismatization $A \supset I = (p)$ $X/_{A/p}$ smooth

$A \to B \rightsquigarrow A/p \to W(A)/p \to W(B)/p$

$X^{\Delta}(B) = X(W(B)/p)$ X^{Δ}: A-algebras $\to$ groupoids

correct for $W(B)$ p-torsion-free.

In general $X^{\Delta}(B) = X(W(B) \overset{L}{\underset{\mathbb{Z}_p}{\otimes}} \mathbb{F}_p) = \underset{B \to W(B) \overset{L}{\underset{\mathbb{Z}_p}{\otimes}} \mathbb{F}_p \text{ in deg } 0}{\operatorname{colim}} X(S)$

Lemma prismatic crystals on X/A $\simeq$ locally free sheaves on X^{Δ} $= \left\{ \begin{array}{l} \mathcal{F}_B \in B\text{-mod} \\ \text{for every } B \text{ with } \operatorname{Spec} W(B)/p \to X \end{array} \right\}$

Pf $\mathcal{E}$ prismatic crystal $\rightsquigarrow \mathcal{F}_B = \mathcal{E}_{W(B)} \otimes_{W(B)} B$ $W(B) \to W(B)/p$ is an object of $(X/A)_{\Delta}$

$\mathcal{F}$ on $X^{\Delta} \rightsquigarrow \mathcal{E}_C = \mathcal{F}_C$ for δ-ring C $\quad \operatorname{Spec} W(C)/p \to \operatorname{Spec} C/p \to X$ ∎

Key lemma v.2 $\mathbb{G}_a^\# = \mathrm{Spf}\, \mathbb{Z}_p\langle x, \frac{x^n}{n!} \mid n\in\mathbb{N}\rangle \longrightarrow \mathbb{G}_a$

$\mathbb{G}_a/\mathbb{G}_a^\# \simeq W/p$ as functors from $\mathbb{Z}_p$-algebras to simplicial rings.

Lemma $\forall R/\mathbb{Z}_p$

$$\ker(F: W(R) \longrightarrow W(R)) \to R$$
$$\wr\!\!\wr$$
$$\mathbb{G}_a^\#(R) \to R$$

Rmk

$$F([a_0] + V[a_1] + \dots) = [a_0^p + pa_1] + V[\dots] + \dots$$

Pf of key lemma v2

$$\begin{array}{ccccc} \mathbb{G}_a^\# \to & W & \xrightarrow{F} & W \\ & V\uparrow & & \uparrow p \\ & W & = & W \end{array} \quad \rightsquigarrow \quad \mathbb{G}_a \xrightarrow{F} W/p$$

with kernel $\simeq \mathbb{G}_a^\#$ ∎

Assume $A = \mathbb{Z}_p$ $X/\mathbb{F}_p$

$$X^\Delta(B) = X((\mathbb{G}_a/\mathbb{G}_a^\#)(B)) \qquad \tilde{X} \longrightarrow X^\Delta$$

$\rightsquigarrow$ if $\tilde{X}$ is a lift of X

$$X^\Delta \simeq \mathrm{colim}\left(\tilde{X} \leftleftarrows (\tilde{X}^{\times 2})^{PD}_\Delta \leftleftarrows (\tilde{X}^{\times 3})^{PD}_\Delta \dots\right)$$

Dwork's lemma: A is a δ-ring $\rightsquigarrow$

$$A \longrightarrow W(A)$$
$$a \longmapsto (a, \delta(a), \delta_2(a), \dots)$$

s.t. $\varphi^n(a) = a^{p^n} + p\,\delta(a)^{p^{n-1}} + \dots + p^n \delta_n(a)$