

Def Prism is a δ -ring A with Cartier divisor $I \subset A$

(Zariski locally on A)

$I=(d)$ for a non zero-div. $d \in A$

s.t.

1) A is (p, I) -complete

2) $p \in I + \varphi_A(I)$

(if $I=(d)$, equivalently, $\delta(d)$ is a unit)

$$\varphi_A(x) = x^p + p\delta(x)$$

δ -structure
 $A \xrightarrow{\delta} A$

\leftrightarrow

ring map

$$A \xrightarrow{\delta} W_2(A)$$

$$a \mapsto (a, \delta(a))$$

splitting $W_2(A) \rightarrow A$

\rightsquigarrow

δ -ring map

$$A \rightarrow W(A)$$

$$a \mapsto (a, \delta(a), \delta_2(a), \dots)$$

$X/A/I$ p -adic formal scheme

prismatic site

$$(X/A)_{\Delta} = \left\{ \begin{array}{c} \delta\text{-ring map} \\ \begin{array}{ccc} B & \rightarrow & B/I_B \text{ with} \\ \uparrow & & \uparrow \\ A & \rightarrow & A/I \end{array} \\ \text{Spf } B/I_B \rightarrow X \\ \downarrow \quad \downarrow \\ \text{Spf } A/I \end{array} \right\}$$

$A \rightarrow B$ is a cover if it is (p, I) -complete flat.

$$\mathcal{O}_{\Delta}(B) = B$$

For affine $X = \text{Spf } R$

$$R\Gamma_{\Delta}(R/A) = \lim_{B \in (R/A)_{\Delta}} B$$

s.t. $\#B > IB$ with bounded p -torsion

Thm 1 $X = \text{Spf } R$ formally smooth $/A/I$

$$\varphi_A^* R\Gamma_{\Delta}(X/A) \otimes_A A/I \cong R\Gamma_{\Delta R}(X/A/I)$$

Ex $A = \mathbb{Z}_p[[u]]$ $A/I = \mathcal{O}_K$ $K \supset \mathbb{Q}_p$ finite p -ext'n

$\forall X/\mathcal{O}_K$ smooth $R\Gamma_{\Delta R}(X/\mathcal{O}_K)$ descends along $\mathbb{Z}_p[[u]] \xrightarrow{u \mapsto u^p} \mathbb{Z}_p[[u]] \rightarrow \mathcal{O}_K$

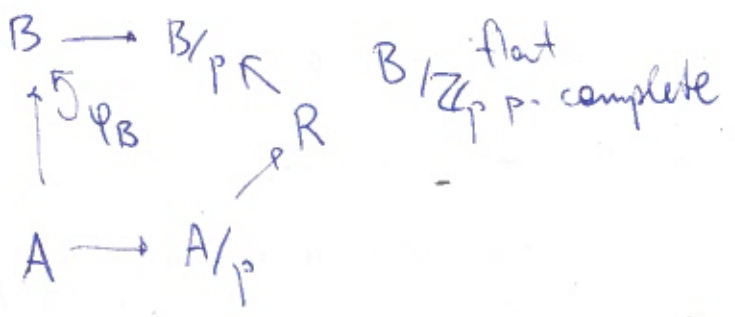
for $k = \mathbb{Q}_p(p^{1/p})$ this implies that $R\Gamma_{\text{dR}}(X/O_k)$ descends to \mathbb{Z}_p .

Pf Assume $I = (p) \subset A$ $R/A/p$ smooth

Will prove $\varphi_A^* R\Gamma_{\Delta}(R/A) \simeq R\Gamma_{\text{cris}}(R/A) \simeq R\Gamma_{\text{dR}}(\hat{R}/A)$

$$\varphi_{\Delta} = \varphi_R^* \quad \text{if } \hat{R}/A \text{ is a flat lift}$$

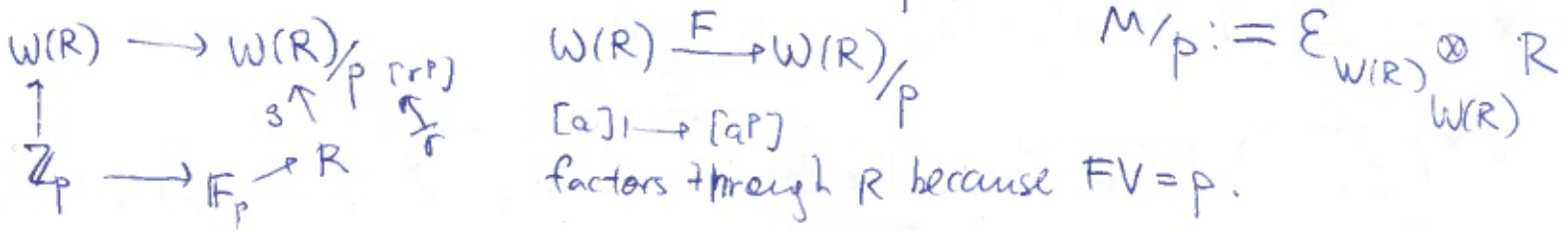
This implies de Rham comparison in general



locally free prismatic crystals on $(R/A)_{\Delta} = \left\{ \begin{array}{l} \text{projective } B\text{-module } \mathcal{E}_B \\ \text{for every } B + \\ \mathcal{E}_B \otimes_B B' \simeq \mathcal{E}_{B'} \\ \text{compatible with composition} \end{array} \right\}$

Prop R/\mathbb{F}_p
 $A = \mathbb{Z}_p$ prismatic crystals on $R/\mathbb{Z}_p =$ crystals on $R =$ projective \hat{R} -modules M
 with $\triangleright: M \rightarrow M \otimes_{\hat{R}} \Omega_{\hat{R}/\mathbb{Z}_p}$
 s.t. \triangleright mod p has loc. nilp. p -curvature

Pf Given \mathcal{E} , how to recover M/p ?



If \tilde{R}/\mathbb{Z}_p is a lift with Frobenius $\varphi_{\tilde{R}} \subset \tilde{R}$ then 3/

$$\begin{array}{ccc} \tilde{R} & \rightarrow & R \\ \uparrow & & \uparrow \\ \mathbb{Z}_p & \rightarrow & \mathbb{F}_p \end{array} \quad M := \frac{1}{p} \varphi_{\tilde{R}}^* E_{\tilde{R}} \quad \text{consistent with above formula}$$

for M/p

$$\begin{array}{ccccc} & & W(\tilde{R}) & \rightarrow & W(R) & \rightarrow & W(R)/p \\ & & \uparrow \delta_{\tilde{R}} & & \uparrow \delta_R & & \uparrow \delta_{R \bmod p} \\ \tilde{R} & \rightarrow & R & \rightarrow & R & \rightarrow & R \\ & & \parallel & & \parallel & & \parallel \\ & & R & & R & & R \end{array}$$

Recall $\left\{ \begin{array}{l} \text{modules } M/\tilde{R} \\ \text{with flat} \\ \text{connection} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{modules } N/\tilde{R} \\ \text{with } p\text{-connection} \end{array} \right\}$

$(\varphi_{\tilde{R}}^* N, \frac{\varphi_{\tilde{R}}^* \nabla}{p}) \leftarrow (N, \nabla)$ $\nabla(fn) = f \nabla n + p d(f) n$

Assume $R = \mathbb{F}_p[x]$ $\tilde{R} = \mathbb{Z}_p[x]$ $\varphi(x) = x^p$

need to produce a connection on

$$\mathbb{Z}_p[x] \rightarrow \mathbb{F}_p[x] \quad \varphi_{\tilde{R}}^* E_{\tilde{R}} =: M$$

Recall crystal structure on $V/\mathbb{Z}_p[x] = p_1^* V \cong p_2^* V$ or

$$\begin{array}{ccc} \mathbb{Z}_p[x_1, x_2] \left[\frac{x_1 - x_2}{p} \right] & \rightarrow & \mathbb{F}_p[x_1, x_2] \left[\frac{x_1 - x_2}{p} \right] \\ \uparrow \varphi & & \uparrow \\ \mathbb{Z}_p[x_1, x_2] & \rightarrow & \mathbb{F}_p[x_1, x_2] \\ \uparrow \varphi_2 & & \uparrow \varphi_1 \\ \mathbb{Z}_p[x] & \rightarrow & \mathbb{F}_p[x] \end{array}$$

$\frac{x_1 - x_2}{p} \mapsto \frac{x_1^p - x_2^p}{p} = \frac{x_1 - x_2}{p} \cdot (\dots)$ for $x \in A$

$A \left[\frac{x}{p} \right] :=$ initial δ -ring $A \rightarrow B$ s.t. $x \in pB$.

$\mathbb{Z}_p[x_1, x_2, \frac{(x_1 - x_2)^n}{n!}] (n \in \mathbb{N})$

$\mathbb{Z}_p[x] \xrightarrow{p_1} \mathbb{Z}_p[x_1, x_2] \xrightarrow{p_2} \mathbb{Z}_p[x]$ satisfying cocycle condition.

prismatic crystal structure $\rightsquigarrow \varphi_1^* E_{\mathbb{Z}_p[x]} \cong E_{\mathbb{Z}_p[x_1, x_2] \left[\frac{x_1 - x_2}{p} \right]} \xrightarrow{\varphi_2^*} E_{\mathbb{Z}_p[x]}$ (*)

Base changing along $\mathbb{Z}_p[x_1, x_2] \left[\frac{x_1 - x_2}{p} \right] \xrightarrow{x_1 \mapsto x_1^p} \mathbb{Z}_p[x_1, x_2] \left[\frac{x_1^p - x_2^p}{p} \right]$

Key Lemma A/\mathbb{Z}_p flat $\varphi_A \circ A$ ~~For some~~ $x \in A$ ~~suppose~~ $x^p \in pA$ 4/
 then $\forall n x^n \in n!A$

Pf Equivalently, need $x^{pk} \in p^{k-1}A$ Will do $k=2$.

$$x^p \in pA \iff \varphi_A(x) \in pA.$$

$$\left(\frac{x^p}{p}\right)^p = \frac{x^{p^2}}{p^p} \equiv \varphi_A\left(\frac{x^p}{p}\right) = \frac{\varphi_A(x^p)^p}{p} = \frac{(x^p + p\delta(x))^p}{p}$$

divisible by p^{p-1}

$$\implies x^{p^2} \in p^{p+1}A. \quad \square$$

Therefore

$$\mathbb{Z}_p\langle x_1, x_2 \rangle \left\langle \frac{x_1^p - x_2^p}{p} \right\rangle \subset \mathbb{Z}_p\langle x_1, x_2, \frac{(x_1 - x_2)^p}{p!} \rangle$$

is equality.

$$\mathbb{Z}_p\langle x_1, x_2 \rangle \left\langle \frac{(x_1 - x_2)^p}{p} \right\rangle$$

and base change of (*) gives $p_1^* M \cong p_2^* M. \quad \square$

Prismatization $A \supset I = (p)$ $X/A/p$ smooth

$$A \rightarrow B \rightsquigarrow A/p \rightarrow W(A)/p \rightarrow W(B)/p$$

$$X^\Delta(B) = X(W(B)/p)$$

correct for $W(B)$ p -torsion-free.

X^Δ : A -algebras \rightarrow groupoids

In general $X^\Delta(B) = X(W(B) \otimes_{\mathbb{Z}_p} \mathbb{F}_p) =$

= colimit $X(S)$

$$S \rightarrow W(B) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$$

in alg

Lemma prismatic crystals \cong locally free sheaves on X/A

$$= \left\{ \mathcal{F}_B \in B\text{-mod} \mid \text{for every } B \text{ with } \text{Spec } W(B)/p \rightarrow X \right\}$$

Pf \mathcal{E} prismatic crystal \rightsquigarrow

$$\mathcal{F}_B = \mathcal{E}_{W(B)} \otimes_{W(B)} B$$

$W(B) \rightarrow W(B)/p$ is an object of $(X/A)/\mathbb{Z}_p$

\mathcal{F} on $X^\Delta \rightsquigarrow \mathcal{E}_C = \mathcal{F}_{W(C)}$ for δ -rim C

$$\text{Spec } C/p \rightarrow \text{Spec } C/p \rightarrow X$$

\square

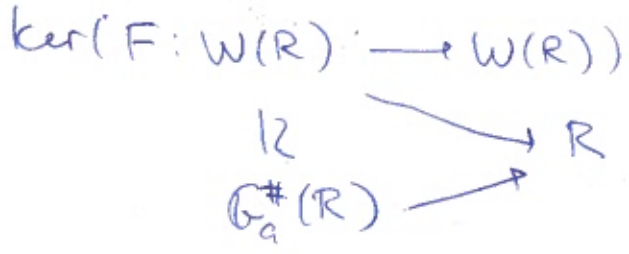
Key lemma v.2

$$\mathbb{G}_a^\# = \text{Spf } \mathbb{Z}_p \langle x, \frac{x^n}{n!} \mid n \in \mathbb{N} \rangle \rightarrow \mathbb{G}_a$$

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$\mathbb{G}_a / \mathbb{G}_a^\# \cong W/p$ as functors from \mathbb{Z}_p -algebras to simplicial rings.

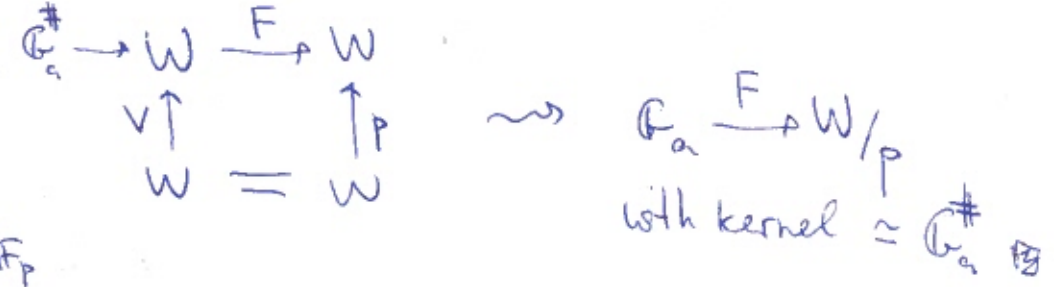
Lemma $\forall R/\mathbb{Z}_p$



Proof

$$F([a_0] + V[a_1] + \dots) = [a_0^p + p a_1] + V[\dots] + \dots$$

Pf of key lemma v.2



Assume $A = \mathbb{Z}_p$ X/\mathbb{F}_p

$$X^\Delta(B) = X((\mathbb{G}_a / \mathbb{G}_a^\#)(B)) \quad \tilde{X} \rightarrow X^\Delta$$

\rightsquigarrow if \tilde{X} is a lift of X

$$X^\Delta \cong \text{colim} \left(\tilde{X} \leftarrow \left(\tilde{X}^{x^2} \right)_{\Delta}^{\text{PD}} \leftarrow \left(\tilde{X}^{x^3} \right)_{\Delta}^{\text{PD}} \dots \right)$$

Dwork's lemma: A is a δ -ring \rightsquigarrow

$$A \rightarrow W(A)$$

$$a \mapsto (a, \delta(a), \delta_2(a), \dots)$$

s.t. $\varphi^n(a) = a^{p^n} + p \delta(a)^{p^{n-1}} + \dots + p^{n-2} \delta_{n-2}(a)$