de Rham cohomology of p-adic formal schemes: a stacky approach

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November 2024

In addition to Bhatt's lecture notes, I found Drinfeld's paper on prismatization and Li–Mondal's paper very helpful.

1 Some clarifications

In the first two sections we work over a characteristic 0 field k.

1.1 Theorem. *Let X be a smooth scheme over k. Let $p : X^{\mathrm{dR},+} \to \mathbb{A}^1/\mathbb{G}_m$ be the structure morphism. Then $Rp_*\mathcal{O}_{X^{\mathrm{dR},+}} \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$ identifies canonically with the Hodge-filtered de Rham cohomology of X.*

Let us first recall some definitions and results from Kenta's talk.

1.2 Definition. Let $\mathbb{G}_a^{\mathrm{dR}}$ be the étale sheaf on k-algebras sending $R \mapsto R_{\mathrm{red}} = R/\operatorname{Nil}(R)$. For a k-scheme X, let X^{dR} be the étale sheaf sending $R \mapsto X(\mathbb{G}_a^{\mathrm{dR}}(R)) = X(R_{\mathrm{red}})$.

When X is smooth, the natural map $X \to X^{\mathrm{dR}}$ is a surjection of presheaves, and one should think of X^{dR} as a quotient of X by the equivalence relation of being infinitesimally close, in other words it is the coequalizer $(X \times_k X)^{\wedge}_{\Delta(X)} \rightrightarrows X$. The category of quasi-coherent sheaves on X^{dR} can be identified with the category of crystals on the infinitesimal site of X, and also with the category of quasi-coherent D_X-modules.

1.3 Definition. Let $\mathbb{G}_a^{\mathrm{dR},+}$ be the stack over $\mathbb{A}^1/\mathbb{G}_m$ given by sending an R-point (L, t) of $\mathbb{A}^1/\mathbb{G}_m$, where L is an invertible R-module and $t : L \to R$ an R-linear map, to the groupoid

$$\operatorname{Cone}(\operatorname{Nil}(R) \otimes_R L \xrightarrow{t} R).$$

Over the open substack $\mathbb{G}_m/\mathbb{G}_m \hookrightarrow \mathbb{A}^1/\mathbb{G}_m$, this recovers $\mathbb{G}_a^{\mathrm{dR},+}$, because there t is an isomorphism and the cone is just $R/\operatorname{Nil}(R) = R_{\mathrm{red}}$.

Over the closed substack $*/\mathbb{G}_m$, where t is zero, this cone is just a split square-zero extension: it ($\mathbb{G}_a^{\mathrm{Hodge}}$) takes the line bundle L to $R \oplus (\operatorname{Nil}(R) \otimes_R L)[1]$.

For a k-scheme X we define a stack $X^{\mathrm{dR},+}$ over $\mathbb{A}^1/\mathbb{G}_m$ sending (L, t) to $X(\mathbb{G}_a^{\mathrm{dR},+}(L, t))$. Similarly we can define $X^{\mathrm{Hodge}}(L) = X(\mathbb{G}_a^{\mathrm{Hodge}}(L))$.

1.4 Remark. In general this process of taking a stack A valued in animated k-algebras (over some base S), and for any k-scheme X associating the stack X^A by $X^A(T \to S) = X(A(T \to S))$, is called *transmutation*.

Ring groupoids

To explain this more explicitly, this cone can be thought of as either a groupoid object in rings, or a dga concentrated in degrees 0 and -1, or a 1-truncated simplicial ring. More generally we have the following definition due to Drinfeld.

1.5 Definition. Let A be a ring. A *quasi-ideal* is an A-module I together with an A-linear map $d: I \to A$, such that for any $x, y \in I$, $d(x)y = d(y)x$.

The data of a quasi-ideal is equivalent to that of a dga

$$\dots 0 \to I \xrightarrow{d} A \to 0 \dots$$

in degrees 0 and -1, and the nontrivial relation is the Leibniz rule.

This data is also equivalent to that of that of an 1-truncated simplicial ring

$$A_1 \leftleftarrows\!\!\!\rightarrow A_0$$

such that $\ker(d_0) \cdot \ker(d_1) = 0$. We can take $A_0 = A$ and $A_1 = A \oplus I$, where A_1 is a ring by $(a, x)(b, y) = (ab, ax + by + xd(y))$, and $d_0(a, x) = a$, $d_1(a, x) = a + d(x)$, and $s(a) = (a, 0)$.

This data is also equivalent to that of a groupoid object in rings. The only missing information we need are a composition map $m : A_1 \times_{A_0} A_1 \to A_1$ and an involution $c : A_1 \to A_1$. By definition m has to be a homomorphism of rings. In particular it has to be a map of abelian groups, which already tells us that it has to send $(f, g) \mapsto f + g - s(a)$, where $a = d_1(f) = d_0(g)$. Then it is a ring homomorphism iff $\ker(d_0) \cdot \ker(d_1) = 0$. Similarly the involution has to send $f \mapsto c(f) = s(d_0(f)) + s(d_1(f)) - f$. This groupoid object in rings is called $\mathrm{Cone}(d)$. When we're in a site (e.g. fpqc) we often want to stackify the prestack $\mathrm{Cone}(d)$, which we will also denote abusively by the same notation.

The last thing I need to say about ring groupoids is how to write down the functor of points shaped like $\mathrm{Cone}(d)$. Let R be another ring, then the groupoid of maps $R \to \mathrm{Cone}(d : I \to A)$ in the $(2, 1)$-category of ring groupoids is given by commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[dr, "d"'] & \overline{R} \arrow[r] \arrow[d, "f"] & R \arrow[r] & 0 \\ & & A & & \end{tikzcd}$$

where the upper row is a ring extension and f is a ring map.

In the situation of $\mathbb{G}_a^{\mathrm{dR},+}$, the map $\mathrm{Nil}(R) \otimes_R L \to R$ is a quasi-ideal, because L is a line bundle. The groupoid of maps from its cone to an arbitrary scheme can thus be modeled in an elementary way by the ring groupoid viewpoint. From this it is clear there is a natural map $X \times \mathbb{A}^1/\mathbb{G}_m \to X^{\mathrm{dR},+}$.

1.6 Exercise. When X is smooth, the natural map $X \times \mathbb{A}^1/\mathbb{G}_m \to X^{\mathrm{dR},+}$ is an étale cover.

Computing pushforwards

Recall also that two talks ago we mentioned a way to compute cohomology of representations of formal completions of vector bundles along the zero section.

The simplest situation is when the vector bundle is just a trivial line bundle over a point. Then there is an equivalence of presentably symmetric monoidal ∞-categories

$$\Phi : \mathcal{D}_{qc}(B\widehat{\mathbb{G}}_a) \simeq \mathcal{D}_{qc}(\mathbb{G}_a)$$

which commutes with the forgetful functors to $\mathcal{D}_{qc}(k)$ on both sides, and takes $\mathcal{O}_{B\widehat{\mathbb{G}}_a}$ to $k = k[t]/t$. So, for a $\widehat{\mathbb{G}}_a$-representation M on the left side, we can compute its cohomology as

$$R\Gamma(B\widehat{\mathbb{G}}_a; M) = R\operatorname{Hom}(\mathcal{O}_{B\widehat{\mathbb{G}}_a}, M) = R\operatorname{Hom}(k, \Phi(M)) = \mathrm{fib}(\Phi(M) \xrightarrow{t} \Phi(M)),$$

which is concentrated in degrees 0 and 1.

The next situation is for a general projective module E over a commutative $\mathbb{Q}$-algebra R. Let $V(E) = \mathit{Spec}(\mathrm{Sym}^*(E^\vee))$ be the associated vector bundle over $\operatorname{Spec} R$. Let $G = V(E)^\wedge_0$, which is a ind-group scheme over $\operatorname{Spec} R$ representing the functor sending an R-algebra S to the S-module $E \otimes_R \mathrm{Nil}(S)$. Then there is an equivalence

$$\Phi : \mathcal{D}_{qc}(BG) \simeq \mathcal{D}_{qc}(V(E^\vee))$$

which commutes with pullback to $\operatorname{Spec} R$ on the left, and pushforward to $\operatorname{Spec} R$ on the right, and takes $\mathcal{O}$ to the module R where $\operatorname{Sym} E$ acts trivially. So, for a G-representation M on the left side, we can compute its cohomology as

$$R\Gamma(BG; M) = R\operatorname{Hom}(\mathcal{O}, M) = R\operatorname{Hom}_{\mathrm{Sym}(E)}(R, \Phi(M)),$$

which can be computed by the Koszul resolution.

Finally we will use the relative situation. Let $f : Y \to Z$ be qcqs morphism of $\mathbb{Q}$-schemes, so that f_* preserves quasicoherence. Let L be a line bundle on Z, and $G = V(L)^\wedge_0$. Suppose G acts on Y in a Z-linear way, so that there is a map $\pi : Y/G \to Z$ (quotient is stacky). Then $R\pi_*$ preserves quasicoherence (using the fiber sequence above).

2 Proof of the theorem

NOTE: this section might not make much sense right now, I'll try to clarify this in a future upload.

In this section X/k is smooth. Kenta stated the following lemmas last time:

2.1 Lemma. *The functor $X \mapsto X^{\mathrm{dR},+}$ commutes with products.*

2.2 Lemma. *If $f : U \to X$ is étale, then*

$$\begin{tikzcd} U \times \mathbb{A}^1/\mathbb{G}_m \arrow[r] \arrow[d] & U^{\mathrm{dR},+} \arrow[d] \\ X \times \mathbb{A}^1/\mathbb{G}_m \arrow[r] & X^{\mathrm{dR},+} \end{tikzcd}$$

is a pullback (in the category of stacks). Consequently by étale descent along the horizontal map we have that $U^{\mathrm{dR},+} \to X^{\mathrm{dR},+}$ is étale (and are open immersions if f is an open immersion).

The proof of this is just by using the infinitesimal lifting property.

2.3 Corollary. *If $X \simeq \operatorname{colim} U$ is a finite colimit of open affines, then $X^{\mathrm{dR},+} \simeq \operatorname{colim} U^{\mathrm{dR},+}$.*

Now we begin the proof. The first step is to show that the pushforward $Rp_*\mathcal{O}_{X^{\mathrm{dR},+}}$ is quasicoherent, and that restricting it to $B\mathbb{G}_m$ is equivalent to pushing forward along $X^{\mathrm{Hodge}} \to B\mathbb{G}_m$. It suffices (why?) to check this locally on X, so we can assume that there is an étale chart $X \to \mathbb{A}^n$. Then the commutative square of stacks over $\mathbb{A}^1/\mathbb{G}_m$

$$\begin{tikzcd} X \times \mathbb{A}^1/\mathbb{G}_m \arrow[r] \arrow[d] & X^{\mathrm{dR},+} \arrow[d] \\ \mathbb{A}^n \times \mathbb{A}^1/\mathbb{G}_m \arrow[r] & (\mathbb{A}^n)^{\mathrm{dR},+} \end{tikzcd}$$

is a pullback. We claim that the bottom map is a torsor over G^n where $G = V(\mathcal{O}(-1))^\wedge_0$ is a sheaf of groups over $\mathbb{A}^1/\mathbb{G}_m$: this follows from the case $n = 1$ which is just the definition. Thus so is the upper horizontal map. Now, this fits into the discussion in the last section which says that pushforward along $X^{\mathrm{dR},+} = (X \times \mathbb{A}^1/\mathbb{G}_m)/G \to \mathbb{A}^1/\mathbb{G}_m$ preserves quasicoherence.

The next step is to identify X^{Hodge}. Consider the canonical map $X^{\mathrm{Hodge}} \to X$. On functors of points for an R-point L of $B\mathbb{G}_m$, it is

$$X(R \oplus M) \to X(R)$$

where $M = (\mathrm{Nil}(R) \otimes_R L)[1]$ and the left hand side is a split square-zero extension. By deformation theory, the fiber above any point $\eta \in X(R)$ is a torsor for $\mathrm{Map}(\mathbb{L}_{X/k}, \eta_* M) = \mathrm{Map}(\eta^* \mathbb{L}_{X/k}, M) = \eta^* T_{X/k} \otimes_R \mathrm{Nil}(R) \otimes_R L$. This is a trivial torsor because it is a split extension. In other words

$$X^{\mathrm{Hodge}} \simeq B(V(T_{X/k} \boxtimes \mathcal{O}(-1))^{\wedge}_0)$$

as stacks over $X \times B\mathbb{G}_m$.

Under the equivalence $\mathcal{D}_{qc}(BV(E)^{\wedge}_0) \simeq \mathcal{D}_{qc}(V(E^\vee))$, we can compute by the Koszul complex

$$Rp_* \mathcal{O}_{X^{\mathrm{Hodge}}} = R\,\mathrm{Hom}_{\mathrm{Sym}^*(\Omega_{X/k}(1))}(\mathcal{O}_{X \times B\mathbb{G}_m}, \mathcal{O}_{X \times B\mathbb{G}_m}) = \bigoplus_i R\Gamma(X, \Omega^i_{X/k}[-i])(i),$$

where the differentials are all zero because $t = 0$.

So we've shown that the assignment $U \mapsto Rp_* \mathcal{O}_{U^{\mathrm{dR},+}}$ is a Zariski sheaf valued in $\mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$, which lies in the heart of the Beilinson t-structure (because we've identified the associated graded and at each filtration step it is concentrated in the correct grading). Thus, this is a sheaf on X valued in chain complexes, equivalently a chain complex of sheaves, and is represented by some chain complex of the form

$$\mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \to \dots$$

It remains to identify the differential. Again it suffices to show this when $X = \mathbb{A}^1$, which should ultimately follow from the explicit identification of $k[t]$-modules with $k[[x]]$-comodules by having t act on $k[[x]]$ by d/dx.

3 Linear algebra in general characteristic

We saw from the above proof that the crucial input was the equivalence of categories

$$\mathcal{D}_{qc}(B\widehat{\mathbb{G}}_a) \simeq \mathcal{D}_{qc}(\mathbb{G}_a)$$

over a field k of characteristic 0. Recall that:

- A $k[t]$-module M on the right side corresponds to the comodule $M \to M[[x]]$, $m \mapsto \exp(xt)m$.
- The equivalence commutes with pullback to $\mathrm{Spec}\, k$ on the left and pushforward to $\mathrm{Spec}\, k$ on the right.
- The equivalence is symmetric monoidal with respect to the standard $\otimes$ on the left and the convolution $\otimes$ on the right.
- The equivalence is compatible with the standard t-structure on the left and the *torsion t-structure* on the right.

As we know from crystalline cohomology, a good replacement for the formal neighborhood of $0 \in \mathbb{A}^1$ is the divided power envelope. Let $\mathbb{G}_a^\sharp = \mathrm{Spec}\, \mathbb{Z}[t, \frac{t^2}{2!}, \frac{t^3}{3!}, \dots]$, which is a group scheme sending any ring R to the abelian group consisting of elements in R with a compatible system of divided powers. This is a quasi-ideal in $\mathbb{G}_a$: suppose $(x, x^{[2]}, x^{[3]}, \dots)$ and $(y, y^{[2]}, y^{[3]}, \dots)$ are elements in $\mathbb{G}_a^\sharp$, then we have $x^n y^{[n]} = n! x^{[n]} y^{[n]} = x^{[n]} y^n$.

3.1 Proposition. *After base change to* $\mathbb{Z}_{(p)}$, $\mathbb{G}_a^\sharp \simeq W^F = \ker(F : W \to W)$ *as quasi-ideals in* $\mathbb{G}_a$, *where* W *is the* p*-typical Witt scheme.*

Here W^F is a quasi-ideal via the map $W^F \hookrightarrow W \twoheadrightarrow W/VW = \mathbb{G}_a$.

Proof. Recall that the construction $R \mapsto W(R)$ is the right adjoint to the forgetful functor from δ-rings to rings, which is in turn the right adjoint to the free δ-ring functor. Thus

$$W(R) = \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[y], W(R)) = \mathrm{Hom}_{\delta\,\mathrm{Ring}}(\mathbb{Z}\{y\}, W(R)) = \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}\{y\}, R)$$

and we can identify W as $\operatorname{Spec}\mathbb{Z}\{y\}$. The δ-ring $\mathbb{Z}\{y\}$ is, as a ring, $\mathbb{Z}[y_0, y_1, \dots]$, and $\delta(y_i) = y_{i+1}$. The Frobenius lift $\phi(x) = x^p + p\delta(x)$ on $\mathbb{Z}\{y\}$ corresponds to the Frobenius on Witt vectors. Thus the kernel of F is represented by $\mathbb{Z}\{y\}/\phi(\mathbb{Z}\{y\}) = \mathbb{Z}[y_0, y_1, \dots]/(y_0^p + py_1, y_1^p + py_2, \dots)$ (then base changed to $\mathbb{Z}_{(p)}$). On the other hand, $(\mathbb{G}_a^\sharp)_{\mathbb{Z}_p}$ has ring of functions

$$\mathbb{Z}_{(p)}[t, \frac{t^p}{p}, \frac{t^{p^2}}{p^{p+1}}, \frac{t^{p^3}}{p^{p^2+p+1}}, \dots]$$

which is exactly the same ring. ∎

3.2 Proposition. *Let* A *be* p*-nilpotent, then* $R\Gamma_{\mathrm{fpqc}}(\operatorname{Spec} A; \mathbb{G}_a^\sharp)$ *is concentrated in degrees 0 and 1.*

Proof. We observe that the map ϕ above is faithfully flat: it is certainly so after inverting p because then the ghost map would be an isomorphism, and on ghost components ϕ is just a shift map; and it is so after reducing mod p because then we would have the Frobenius on $\mathbb{F}_p[y_0, y_1, \dots]$. This is enough to conclude that ϕ itself is faithfully flat. Thus we have a short exact sequence of group schemes

$$0 \to \mathbb{G}_a^\sharp \to W \xrightarrow{F} W \to 0$$

in the fpqc topology. It suffices to compute $R\Gamma_{\mathrm{fpqc}}(\operatorname{Spec} A; W)$ is concentrated in degree 0. We can write $R\Gamma(\operatorname{Spec} A; W) = R\Gamma(\operatorname{Spec} A; \lim W_n) \simeq R\lim R\Gamma(\operatorname{Spec} A; W_n)$. Because the maps $W_n(A) \to W_{n-1}(A)$ are all surjective, it suffices to show $R\Gamma(\operatorname{Spec} A; W_n) \simeq W_n(A)[0]$ and there are no $\lim^1$ terms. By definition there is a filtration of W_n by $\mathbb{G}_a$'s (Witt vectors with first i coordinates zero), so it suffices to show it for $\mathbb{G}_a$. This is just the classical fpqc descent statement. ∎

3.3 Remark. Bhatt makes a remark here that flat cohomology requires a cutoff cardinal to define in general, but in this case there is no issue. Because H^1 of an abelian sheaf is the same as the isomorphism classes of torsors, this proposition is equivalent to saying that the functor taking a p-nilpotent ring R to the groupoid of torsors $B\mathbb{G}_a^\sharp(R)$ (which by Dold–Kan can be viewed as an object in $\mathcal{D}(\mathbb{Z})$) is a fpqc sheaf.

3.4 Proposition. *We have a natural equivalence*

$$\mathcal{D}_{qc}(B\mathbb{G}_a^\sharp) \simeq \mathcal{D}_{qc}(\widehat{\mathbb{G}_a})$$

which commutes with pullback to $\operatorname{Spec}\mathbb{Z}$ *on the left and local cohomology at 0 on the right.*

Recall that local cohomology is just the colimit of the cohomology of pullbacks to $\operatorname{Spec}\mathbb{Z}[x]/x^n$.

Proof. Let $f : \operatorname{Spec}\mathbb{Z} \to B\mathbb{G}_a^\sharp$. Given an object V on the left, I need to give f^*V a locally nilpotent operator on it. There is a short exact sequence of quasicoherent sheaves on $B\mathbb{G}_a^\sharp$:

$$0 \to \mathcal{O}_{B\mathbb{G}_a^\sharp} \to f_*\mathcal{O}_{\mathbb{Z}} \xrightarrow{N} f_*\mathcal{O}_{\mathbb{Z}} \to 0$$

where N corresponds to the map d/dt on $f_*\mathcal{O}_{\mathbb{Z}} = \mathbb{Z}[t, \frac{t^2}{2!}, \dots]$. Thus we have a cofiber sequence in the derived category,

$$0 \to V \to V \otimes f_*\mathcal{O}_{\mathbb{Z}} \to V \otimes f_*\mathcal{O}_{\mathbb{Z}} \to 0.$$

By the projection formula,

$$R\Gamma(B\mathbb{G}_a^\sharp; V \otimes f_* \mathcal{O}_\mathbb{Z}) = R\operatorname{Hom}(\mathcal{O}_{B\mathbb{G}_a^\sharp}, f_* f^* V) = R\operatorname{Hom}(\mathcal{O}_\mathbb{Z}, f^*V) = f^*V.$$

Thus we get an operator $N_V : f^*V \to f^*V$. To see it is locally nilpotent, it suffices to show that $R\Gamma(B\mathbb{G}_a^\sharp; -)$ commutes with filtered colimits, and by the exact sequence we've reduced to showing $R\Gamma(B\mathbb{G}_a^\sharp; - \otimes f_*\mathcal{O}_\mathbb{Z}) = f^*(-)$ commutes with filtered colimits, which is clear. Thus we can assign V to the pair (f^*V, N_V) which belongs to the right side.

We can give the inverse map just for abelian groups M with a locally nilpotent operator N (and extend to the full derived category by sifted colimits and desuspensions). This goes by making. the coaction

$$M \to M \otimes \mathbb{Z}[t, \frac{t^2}{2!}, \dots]$$

taking $m \mapsto \exp(tN)m = \sum N^i m \cdot \frac{t^i}{i!}$. □

3.5 Remark. Monoidality: let's test this in the simplest case. For M, N abelian groups the coaction on $M \otimes N$ should take $m \otimes n \mapsto \sum_i N^i(m \otimes n) \cdot \frac{t^i}{i!} = \sum_i \sum_j \binom{i}{j} N^j(m) \otimes N^{i-j}(n) \frac{t^i}{i!}$ which is correct.

t-structure: let's define the torsion t-structure...

3.6 Remark. Cohomology of $B\mathbb{G}_a^\sharp$: let $s : \operatorname{Spec}\mathbb{Z} \to \widehat{\mathbb{G}_a}$ be the zero section. Then

$$R\Gamma(B\mathbb{G}_a^\sharp; \mathcal{O}_{B\mathbb{G}_a^\sharp}) = R\operatorname{Hom}_{\widehat{\mathbb{G}_a}}(s_*\mathcal{O}_\mathbb{Z}, s_*\mathcal{O}_\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}[-1]$$

using the resolution

$$0 \to \mathbb{Z}[[t]] \xrightarrow{t} \mathbb{Z}[[t]] \to \mathbb{Z} \to 0.$$

3.7 Remark. There is also a version of this in families: suppose E is a locally free finite rank sheaf on a scheme, and $V(E)$ its total space, then define $V(E)^\sharp$ to be the divided power envelope of the zero section. Then we would have an equivalence

$$\mathcal{D}_{qc}(BV(E)^\sharp) \simeq \mathcal{D}_{qc}(\widehat{V(E^\vee)})$$

enjoying similar properties. Using the Koszul resolution we also have

$$R\Gamma(BV(E)^\sharp; \mathcal{O}) = \bigoplus_i \wedge^i E^\vee[-i].$$

4 de Rham stack in positive/mixed characteristic

Throughout, we will let V be a p-complete ring with bounded p^∞-torsion, such as a p-nilpotent ring $\mathbb{F}_p$ or a p-torsion free ring $\mathbb{Z}_p$. The output of the de Rham stack functor will be a p-adic formal stack, which we now briefly explain:

The category $\mathrm{St}_{\mathbb{Z}_p}^\wedge$ of p-adic formal stacks is defined as

$$\mathrm{St}_{\mathbb{Z}_p}^\wedge = \lim \mathrm{St}_{\mathbb{Z}/p^n\mathbb{Z}}$$

For any stack over $\mathbb{Z}_p$ we can thus associate it with a p-adic formal stack. Another way to say the same is that a p-adic formal stack is just one that takes as input p-nilpotent rings.

4.1 Definition. Let $\mathbb{G}_a^{\mathrm{dR}}$ be the ring groupoid $\operatorname{Cone}(\mathbb{G}_a^\sharp \to \mathbb{G}_a)$, stackified in the fpqc topology. By the vanishing result, we see that on functor of points, it takes a p-nilpotent V-algebra R to

$$\mathbb{G}_a^{\mathrm{dR}}(R) = \operatorname{cofib}(B\mathbb{G}_a^\sharp(R)[-1] \to R)$$

which is given by an explicit chain complex whose cohomology is concentrated in degrees $0, 1$.

4.2 Proposition. $\mathbb{G}_a^{\mathrm{dR}} \simeq \mathrm{Cone}(p : W \to W)$.

Proof. By the previous characterization, the map $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ is the same as the map $W^F \to W \to W/VW$. Thus, $\mathrm{Cone}(W^F \to W/VW) = \mathrm{Cone}(VW \to W/W^F) = \mathrm{Cone}(VW \xrightarrow{F} W) = \mathrm{Cone}(W \xrightarrow{FV=p} W)$. $\square$

4.3 Definition. Let $\mathbb{G}_a^{\mathrm{dR},+} \to \mathbb{A}^1/\mathbb{G}_m$ send an R-point (L,t) to

$$\mathrm{cofib}(B\mathbb{G}_a^\sharp(R) \otimes_R L[-1] \xrightarrow{t} R)$$

which recovers $\mathbb{G}_a^{\mathrm{dR}}$ above the open locus. Above the closed locus it is given by the split, square-zero extension

$$\mathbb{G}_a^{\mathrm{Hodge}}(L) = R \oplus B\mathbb{G}_a^\sharp(R) \otimes_R L.$$

4.4 Definition. Let X be a smooth p-adic formal scheme[1] over V. Define $(X/V)^{\mathrm{dR},+} \to \mathbb{A}^1/\mathbb{G}_m$ by taking an R-point of $\mathbb{A}^1/\mathbb{G}_m$ to $X(\mathbb{G}_a^{\mathrm{dR},+}(R))$ in the derived algebraic geometry sense. Similarly we have $(X/V)^{\mathrm{dR}}, (X/V)^{\mathrm{Hodge}}$.

4.5 Proposition. *The set of p-nilpotent rings R such that all $\mathbb{G}_a^\sharp$-torsors are trivial forms a basis of the fpqc topology.*

Bhatt says that this follows from a small object argument. (...)

For such an R, the surjection $R \to \mathbb{G}_a^{\mathrm{dR}}(R) = R/\operatorname{im}(\mathbb{G}_a^\sharp(R))$ has locally nilpotent kernel. In particular it is an universal homeomorphism, thus induces an isomorphism on étale sites. Similarly, on π_0, the map $R \to \mathbb{G}_a^{\mathrm{dR},+}(R)$ is surjective with locally nilpotent kernel, thus induces an isomorphism on étale sites.

4.6 Theorem. *Let X be a smooth p-adic formal scheme over V. Let $p : X^{\mathrm{dR},+} \to \mathbb{A}^1/\mathbb{G}_m$ be the structure morphism. Then $Rp_*\mathcal{O}_{X^{\mathrm{dR},+}} \in \mathcal{D}_{qc}(\mathbb{A}^1/\mathbb{G}_m)$ identifies canonically with the Hodge-filtered de Rham complex of X/V.*

Let's try to adapt the proof in the characteristic 0 case here. The following justify the analogs of the lemmas appearing there:

1. It is still clear that the functor $X \mapsto X^{\mathrm{dR},+}$ commutes with products.
2. The maps $X \to (X/V)^{\mathrm{dR}}$, $X \times \mathbb{A}^1/\mathbb{G}_m \to (X/V)^{\mathrm{dR},+}$ are fpqc covers. Indeed, it suffices to show that $X(R) \to X(S)$ is surjective for $R \to S$ a map of animated p-nilpotent V-algebras which is surjective on π_0 with locally nilpotent kernel. But this is just the smooth lifting criterion.
3. Similarly the pullback square can be checked on rings with no nontrivial $\mathbb{G}_a^\sharp$-torsors.
4. The same argument gives the colimit statement.

The same proof then goes through.

Crystalline cohomology

From this theorem, we obtain a conceptual explanation of the "crystalline miracle":

4.7 Corollary. *Let X be a smooth qcqs p-adic formal scheme over a torsion-free V. Then $R\Gamma(X;\Omega^*_{X/V})$ depends functorially on the V/p-scheme $X_{p=0}$.*

[1] Recall this means a formal scheme which is locally of form $\mathrm{Spf}(B,(p))$ for an V-algebra B.

Proof. It suffices to show that $(X/V)^{\mathrm{dR}}$ depends functorially on $X_{p=0}$. Thus it suffices to show that for any p-nilpotent V-algebra R, $\mathbb{G}_a^{\mathrm{dR}}(R)$ admits a functorial animated V/p-algebra structure. We can further reduce this to showing that $\mathrm{Cone}(\mathbb{G}_a^\sharp(\mathbb{Z}_p) \to \mathbb{Z}_p) = \mathbb{F}_p$, which follows because the only elements in $\mathbb{Z}_p$ which admit divided powers are $p\mathbb{Z}_p$. $\square$

One can in fact define the crystallization of an V/p-scheme X to be the stack

$$(X/V)^{\mathrm{cris}}(R) = \underline{\mathrm{Hom}}_{V/p}(\mathrm{Spec}(\mathbb{G}_a^{\mathrm{dR}}(R)), X),$$

and consider the cohomology of its structural sheaf. One can show formally by the same argument as the main proof that this is an object in $\mathrm{CAlg}(\mathcal{D}(V))_p^\wedge$ which lifts the de Rham complex of $X/(V/p)$, and in fact this is exactly the crystalline cohomology.

4.8 Remark. When V is the Witt vectors of a perfect $\mathbb{F}_p$-field, the above shows that the de Rham complex $M = R\Gamma(X; \Omega^*_{X/V})$ (and in fact, the de Rham stack) admits a Frobenius $\phi_M : \phi^* M \to M$ induced by the relative Frobenius on $X_{p=0}$ over k.